The Nature and Extent of Particles

W.B. (ICN/UNAM) and U.-J. Wiese (Univ. of Bern)

Motivation: Classical (quantum) mechanics: point particles with exact (uncertain) position. How about quantum field theory?
We deal with its covariant formalism, but how does it describe particles?

Classical Hamilton mechanics: point particles with $x(t)$, $p(t)$

Canonical QM: Hermitian operators $\hat{x}$, $\hat{p}$, $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$, $\Delta x \, \Delta p \geq \hbar/2$

Position smeared out to wave function $\Psi(t,x) = \langle x|\psi(t)\rangle$, point particle with uncertain position $\rightarrow$ interference effects

Field theory: degrees of freedom: immobile fields, no particle coordinates, *e.g.* classical electrodynamics $\vec{E}(t,\vec{x})$, $\vec{B}(t,\vec{x})$, or continuum mechanics with density $\rho(t,\vec{x})$. A wave covers some region, not point-like.

Canonical quantum field theory:

Operator-valued fields, excitations $\sim$ "particles", *e.g.* photons in QED, or phonons in field theory of a vibrating crystal, fields fluctuate in quanta.

Terminology (in this talk)

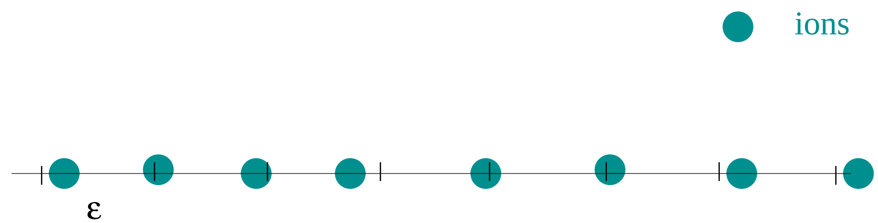
- **"Particle":** QM, position usually uncertain, but point-like
- **"Wavicle":** Quantized field excitation, relativistic and more fundamental. Features and localization are subtle, to be discussed in a toy model.

  Term suggested by Wilczek, distinction unusual, but useful for this talk.

Toy model: 1d chain of coupled ions

Ions described by particles in (non-relativistic) QM

"Crystal": coupled harmonic oscillators, equilibrium distance ϵ



$$\hat{H} = \sum_{n=1}^{N} \left[\frac{1}{2M}\hat{p}_n^2 + \frac{1}{2}M\omega_0^2\left(\hat{x}_{n+1} - \hat{x}_n - \epsilon\right)^2\right]$$

$$= \sum_{n=1}^{N} \left[\frac{1}{2M}\hat{p}_n^2 + \frac{1}{2}M\omega_0^2\left(\hat{y}_{n+1} - \hat{y}_n\right)^2\right], \quad \hat{y}_n = \hat{x}_n - n\epsilon, \ \hat{y}_{N+1} = \hat{y}_1$$

M: ion mass; $\hat{y}_n$: displacement; periodic b.c.: $\hat{x}_{N+1} = \hat{x}_1 + \overbrace{N\epsilon}^{L}$

Actually a ring, simplifies Fourier analysis, at the end N, $L \to \infty$.

Strictly speaking: model should be in space-time dimension > 2 and $L \to \infty$, otherwise no crystalline order: requires spontaneous breaking of translation sym. [Mermin-Wagner Theorem], but 1d notation is instructive.

- $\mathbb{Z}(N)$ shift symmetry: $\hat{y}_n \to \hat{y}_{n+1}$ $(n < N)$, and $\hat{y}_N \to \hat{y}_1$

  Fourier series:

$$\hat{y}(k) = \sum_{n=1}^{N} \hat{y}_n e^{-\mathrm{i}n\epsilon} = \hat{y}(-k)^\dagger = \hat{y}\Big(k + \frac{2\pi}{\epsilon}\Big) \ , \quad \hat{y}_n = \frac{1}{N} \sum_k \hat{y}(k) e^{\mathrm{i}n\epsilon}$$

Bloch wave number $k = \frac{2\pi m}{L} \in B = (-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]$
(actually k_m, but this messes up the notation).

Periodicity over Brillouin zone B due to $\epsilon > 0$, and discrete if $L > \infty$.
For N even: $m = -\frac{N}{2} + 1, -\frac{N}{2} + 1, \ldots \frac{N}{2} - 1, \frac{N}{2}$

- Sym. under continuous translation of all ions.
  Noether: conservation of total momentum $\hat{P} = \sum_n \hat{p}_n$

Hamilton operator in Fourier space

$$\hat{T} = \frac{1}{2M}\sum_n \hat{p}_n^2 = \frac{1}{2MN}\sum_k \hat{p}(k)^\dagger \hat{p}(k) \qquad \text{(Parseval)}$$

$$\hat{V} = \frac{M\omega_0^2}{2}\sum_n (\hat{y}_{n+1} - \hat{y}_n)^2$$

$$= \frac{M\omega_0^2}{2N^2}\sum_{k,\,k'} \hat{y}(k)\hat{y}(k')\Big[\sum_n \left(e^{-\mathrm{i}k(n+1)\epsilon} - e^{-\mathrm{i}kn\epsilon}\right)\left(e^{-\mathrm{i}k'(n+1)\epsilon} - e^{-\mathrm{i}k'n\epsilon}\right)\Big]$$

$$= \frac{M}{2N}\sum_k \hat{y}(k)^\dagger \hat{y}(k)\,\omega(k)^2\,, \qquad \omega(k) = 2\omega_0\left|\sin\frac{k\epsilon}{2}\right|$$

where we used Poisson's formula: $\frac{1}{N}\sum_n e^{\mathrm{i}(k-k')n\epsilon} = \delta_{kk'}$

$$\hat{H} = \hat{T} + \hat{V} = \frac{\hat{P}^2}{2MN} + \frac{1}{N}\sum_{k\neq 0}\left[\frac{\hat{p}(k)^\dagger \hat{p}(k)}{2M} + \frac{M\omega(k)^2}{2}\hat{y}(k)^\dagger \hat{y}(k)\right]$$

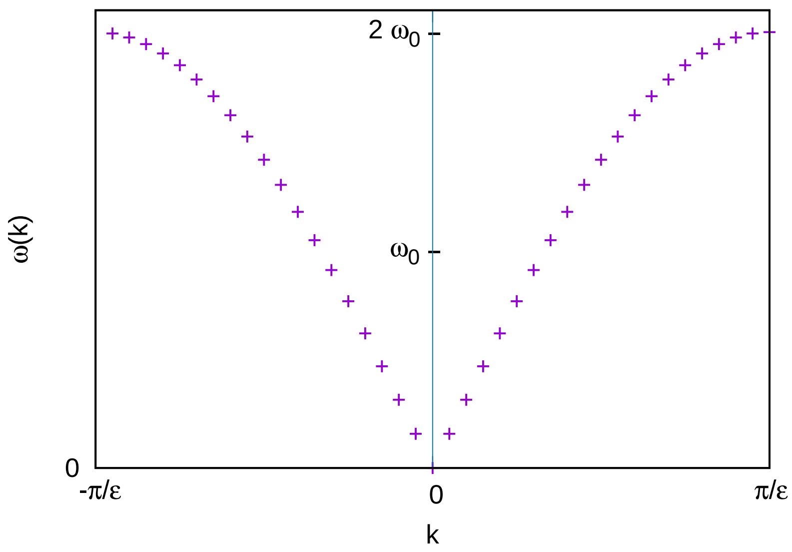
The $k = 0$-mode represents an overall motion, with $\hat{P}$ and mass MN.

$k \neq 0$ (discrete, in B) oscillations with $\omega(k)$

$\omega(k) = 2\omega_0 \left|\sin\frac{k\epsilon}{2}\right|$ for $N = 40$:

$|k|$ small: $\omega \simeq \omega_0|k|\epsilon$,

$|k| \lesssim \pi/\epsilon$: $\omega \lesssim 2\omega_0$



Phonons: quantized vibrations

Phonon annihilation and creation operator, analogous to harm. oscillator

$$\hat{a}(k) = \frac{1}{\sqrt{2N}}\Big(\alpha(k)\hat{y}(k) + \frac{\mathrm{i}}{\hbar\alpha(k)}\hat{p}(k)\Big)$$

$$\hat{a}(k)^\dagger = \frac{1}{\sqrt{2N}}\Big(\alpha(k)\hat{y}(k)^\dagger - \frac{\mathrm{i}}{\hbar\alpha(k)}\hat{p}(k)^\dagger\Big)$$

$$\alpha(k) \doteq \sqrt{M\omega(k)/\hbar} \qquad (\text{for } k \neq 0)$$

Familiar commutation relations

$$[\hat{y}_n, \hat{p}_{n'}] = \mathrm{i}\hbar\delta_{nn'} \ , \quad [\hat{y}_n, \hat{y}_{n'}] = 0 = [\hat{p}_n, \hat{p}_{n'}]$$

imply in Fourier space

$$[\hat{y}(k), \hat{p}(k')^\dagger] = \mathrm{i}\hbar N \delta_{kk'} \ , \quad [\hat{y}(k), \hat{y}(k')] = 0 = [\hat{p}(k), \hat{p}(k')]$$

and therefore

$$[\hat{a}(k), \hat{a}(k')^\dagger] = \mathrm{i}\hbar N \delta_{kk'} \ , \quad [\hat{a}(k), \hat{a}(k')] = 0 = [\hat{a}(k)^\dagger, \hat{a}(k'))^\dagger]$$

Computation of $\hat{a}(k)^\dagger \hat{a}(k)$ leads to another familiar expression:

$$\hat{H} = \frac{\hat{P}^2}{2NM} + \sum_{k \neq 0} \hbar\omega(k) \Big(\underbrace{\hat{a}(k)^\dagger \hat{a}(k)}_{\hat{\nu}(k)} + \frac{1}{2} \Big)$$

$\hat{\nu}(k)$: phonon number operator of the Bloch wave number k

Quantum states of a vibrating crystal

Energy eigenstates are given by: • total momentum P
• phonon occupation number $\nu(k)$ to each Bloch wave number

Ground state $|0\rangle$: $P = 0, \ \nu(k) = 0, \ \forall k$

$$\hat{P}|0\rangle = 0 \ , \quad \hat{a}(k)|0\rangle = 0 \ , \quad E_0 = \langle 0|\hat{H}|0\rangle = \frac{\hbar}{2}\sum_k \omega(k)$$

Thermodyn. limit $N \to \infty$: $\sum_k \to \int_B dk$, energy density

$$\frac{E_0}{L} = \frac{\hbar}{4\pi}\int_{-\pi/\epsilon}^{\pi/\epsilon} dk \ 2\omega_0 \left|\sin\frac{k\epsilon}{2}\right| = \frac{2\hbar\omega_0}{\pi\epsilon} \underbrace{\longrightarrow}_{\epsilon\to 0} \infty$$

Cosmological Constant (truncate $\frac{\pi}{\epsilon} = M_{\text{Planck}}$, requires TINY $\omega_0 \ldots$).

Consider only energy <u>differences</u>, but maintain $N \to \infty$

$\frac{P^2}{2MN} = 0$ at any finite P, degenerate ground states

SSB of translation invariance, phonon as Nambu-Goldstone boson (in $d > 2$)

- 1-phonon state: $|k\rangle = \hat{a}(k)^\dagger |0\rangle$

$$E(k) - E_0 = \hbar\omega(k) = 2\hbar\omega_0 \left| \sin \frac{k\epsilon}{2} \right|$$

▷ small $|k|$: $E(k) \simeq \hbar |k| \, \omega_0 \epsilon$
linear in $\hbar k$, $i.e.$ "relativistic" with respect to $c \doteq \omega_0 \epsilon$: speed of sound

▷ $|k| \to \frac{\pi}{\epsilon}$: group velocity $c(k) = \left| \frac{\partial E}{\partial(\hbar k)} \right| = \omega_0 \epsilon \cos \frac{k\epsilon}{2} \to 0$

- 2-phonon state : $|k_1, k_2\rangle = \hat{a}(k_1)^\dagger \hat{a}(k_2)^\dagger |0\rangle = |k_2, k_1\rangle$

Bosons, $\nu(k)$ unlimited

Wavicle localization

How extended are wavicle, in this case 1-phonon states?

QM position eigenstate

$$|n\rangle \;=\; \frac{\epsilon}{2\pi}\int_B dk\,|k\rangle e^{\mathrm{i}kn\epsilon} = \hat{a}_n^\dagger|0\rangle \quad \text{with} \quad \hat{a}_n^\dagger = \frac{\epsilon}{2\pi}\int_B dk\,\hat{a}(k)^\dagger e^{\mathrm{i}kn\epsilon}$$

$$\Rightarrow \qquad [\hat{a}_n, \hat{a}_{n'}^\dagger] = \delta_{nn'}\,, \quad [\hat{a}_n, \hat{a}_{n'}] = 0 = [\hat{a}_n^\dagger, \hat{a}_{n'}^\dagger]$$

QM picture: $\hat{a}_n^\dagger$ creates a phonon, solely at position n.
Works for decoupled oscillators (no geometry!), but fails in field theory:

$$\hat{a}_n^\dagger = \frac{\epsilon}{2\pi}\int_B dk\,\frac{1}{\sqrt{2N}}\underbrace{\left(\alpha(k)\hat{y}(k)^\dagger - \frac{\mathrm{i}}{\hbar\alpha(k)}\hat{p}(k)^\dagger e^{\mathrm{i}kn\epsilon}\right)}_{\hat{a}(k)^\dagger}\,, \quad \alpha(k) \doteq \sqrt{\frac{M\omega(k)}{\hbar}}$$

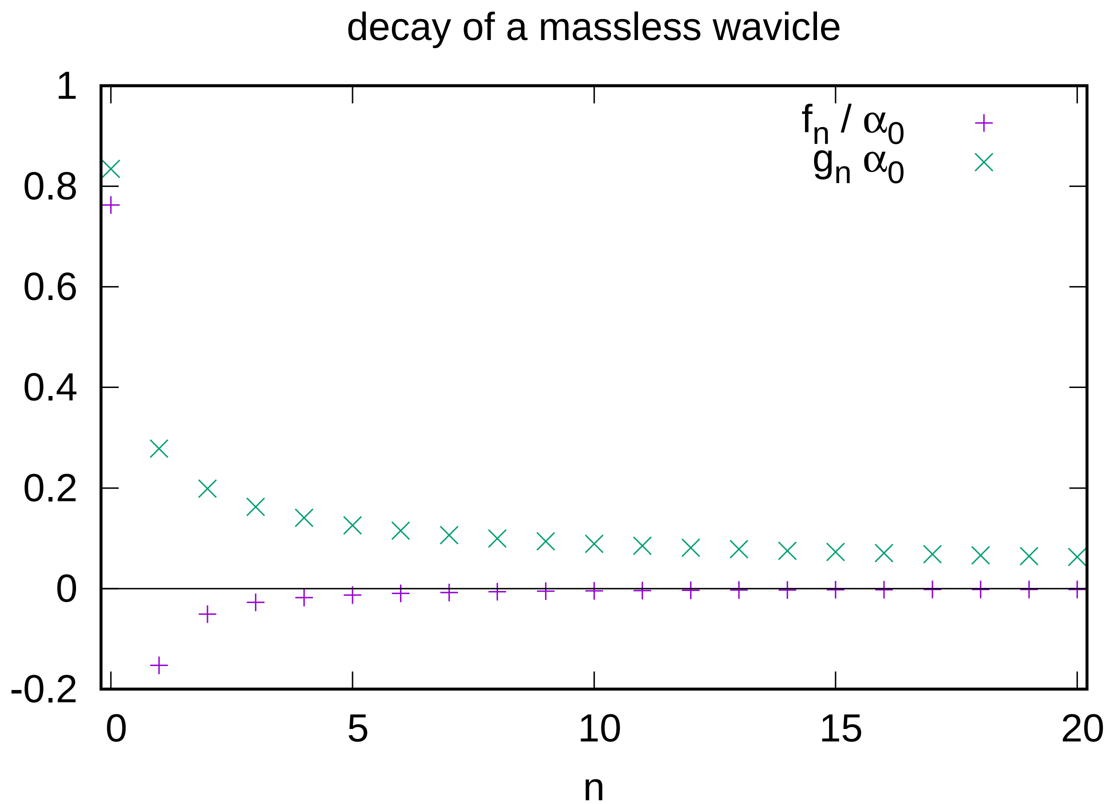
Convolution in coordinate space:

$$
\begin{aligned}
\hat{a}_n^\dagger &= \frac{1}{\sqrt{N}} \sum_{n'} \left(f_{n-n'}\, \hat{y}_n - g_{n-n'}\, \frac{\mathrm{i}}{\hbar} \hat{p}_n \right) \\
f_n = f_{-n} &= \frac{\epsilon}{2\pi\sqrt{\hbar}} \int_B dk\, \left(M\omega_0 \left| \sin\frac{k\epsilon}{2} \right| \right)^{1/2} \cos(kn\epsilon) \\
g_n = g_{-n} &= \frac{\epsilon\sqrt{\hbar}}{2\pi} \int_B dk\, \left(M\omega_0 \left| \sin\frac{k\epsilon}{2} \right| \right)^{-1/2} \cos(kn\epsilon)
\end{aligned}
$$

At large $|n|$:

$$
f_n \propto \frac{1}{|n|^{3/2}}\,, \qquad g_n \propto \frac{1}{|n|^{1/2}}
$$

Power-law decays $\longrightarrow$ **non-local!** Creation of one phonon at position n requires exciting the entire crystal, in a coordinated manner.



Wavicle decay, in units of $\alpha_0 \doteq \sqrt{M\omega_0/\hbar}$. The convolution terms f_n and g_n describe excitations of ions, at distance $n\epsilon$, regarding displacement and momentum. Both are <u>non-local</u>; g_n decays very slowly.

Massive wavicles

So far: phonon mass $= 0$ $\quad$ ($\neq$ ion mass M), NGBs in $d > 2$

Add mass by explicit translation sym. breaking:

$$\hat{V}' = \frac{M{\omega_0}'^2}{2}\sum_n(\overbrace{\hat{x}_n - n\epsilon}^{\hat{y}_n})^2 = \frac{M{\omega_0}'^2}{2N}\sum_k \hat{y}(k)^\dagger \hat{y}(k)$$

$$\hat{H} = \hat{T} + \hat{V} + \hat{V}' = \hbar\sum_k \Big(\hat{\nu}(k) + \frac{1}{2}\Big)\omega(k)$$

$$\omega(k) = \left[\left(2\omega_0 \sin\frac{k\epsilon}{2}\right)^2 + {\omega_0}'^2\right]^{1/2}$$

At small $|k|$: "relativistic"

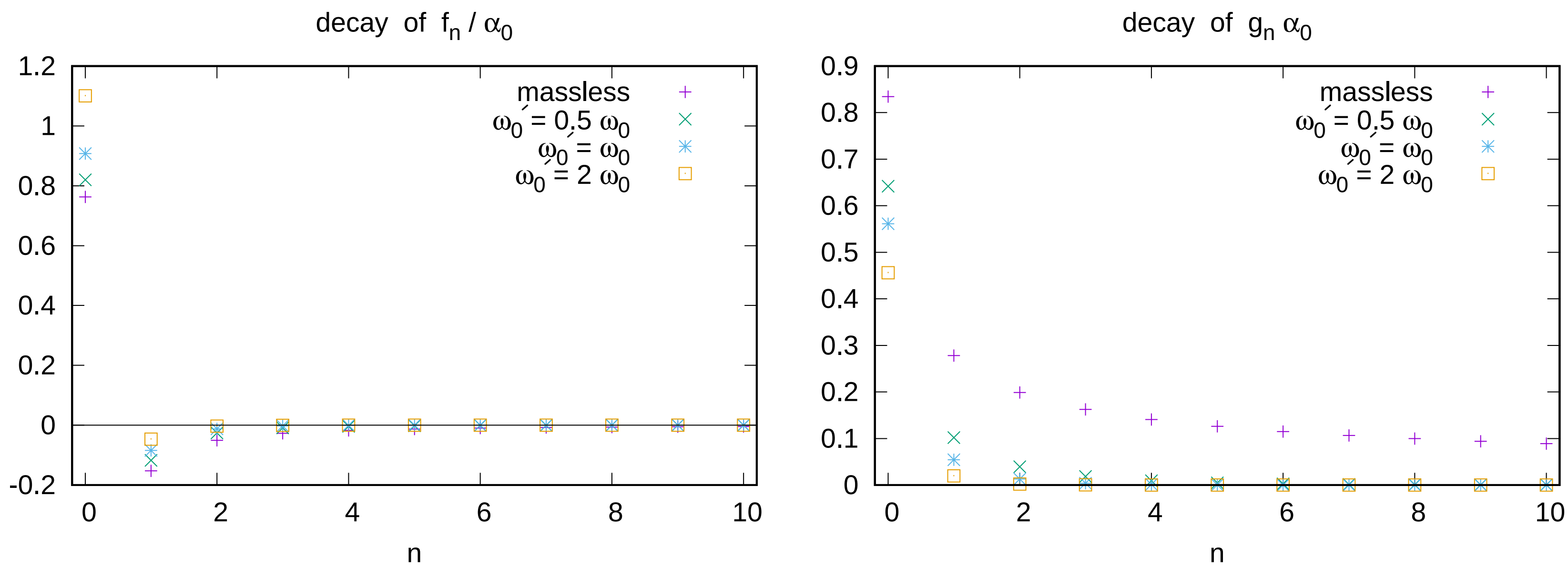
$$E_{\text{phonon}}(k) = \hbar\omega(k) \simeq \Big[(\hbar k c)^2 + (\overbrace{\hbar\omega_0'}^{mc^2})^2\Big]^{1/2}$$

Phonon mass $m = \frac{\hbar\omega_0'}{c^2} = \frac{\hbar\omega_0'}{\omega_0^2\epsilon^2}$ (with $c = \omega_0\epsilon$)

Now $f_n,\ g_n$ decay exponentially:

$$f_n,\ g_n \propto \exp\Big(-\frac{mc\,\overbrace{|n|\epsilon}^{|x|}}{\hbar}\Big) \doteq \exp\Big(-\frac{|x|}{\lambda_{\text{phonon}}}\Big)\ , \quad \lambda_{\text{phonon}} = \frac{\hbar}{mc}$$

Compton wave length, localization scale of a massive phonon, local



Decay of f_n and g_n (in units of α_0), related to ion displacement and momentum, for wavicle mass $m = \hbar\omega_0'/c^2$, with $\omega_0' = 0,\ 0.5\omega_0,\ \omega_0,\ 2\omega_0$.

Transition to field theory, continuum limit

Debye: field theory of vibrating solid $\sim$ lattice regularized quantum field theory

Scalar field and its conjugate:

$$\hat{\Phi}(x = n\epsilon) = \sqrt{M\omega_0^2\epsilon}\;\hat{y}_n$$

$$\hat{\Pi}(x = n\epsilon) = \frac{1}{\sqrt{M\omega_0^2\epsilon}}\,\hat{p}_n = -\frac{i\hbar}{\sqrt{M\omega_0^2\epsilon^3}}\,\frac{\partial}{\partial y_n} = -\frac{i\hbar}{\epsilon}\,\frac{\partial}{\partial\Phi}$$

Constants arrange for right dimensions, and

$$[\hat{\Phi}(x), \hat{\Pi}(x')] = \frac{i\hbar}{\epsilon}\delta_{nn'} \overbrace{\longrightarrow}^{\epsilon\to 0} i\hbar\delta(x - x')$$

$$\hat{H} = \sum_n \left[\frac{\hat{p}_n^2}{2M} + \frac{M\omega_0^2}{2}(\hat{y}_{n+1} - \hat{y}_n)^2 + \frac{M{\omega_0}'^2}{2}\hat{y}_n^2\right]$$

$$= \frac{\epsilon}{2}\sum_x \left[c^2\hat{\Pi}(x)^2 + \left(\frac{\hat{\Phi}(x+\epsilon)}{\epsilon}\right)^2 + \left(\frac{mc}{\hbar}\right)^2\hat{\Phi}(x)^2\right]$$

$$\overbrace{\longrightarrow}^{\epsilon\to 0} \frac{1}{2}\int_0^L dx \left[c^2\hat{\Pi}(x)^2 + \partial_x\hat{\Phi}(x)\partial_x\hat{\Phi}(x) + \left(\frac{mc}{\hbar}\right)^2\hat{\Phi}(x)^2\right]$$

Continuum limit: renormalize ω_0 such that $c = \omega_0\epsilon$ is kept finite, can be turned into the speed of light $\rightarrow$ covariant field theory

Then $\lambda_\Phi^{-1} = \frac{mc}{\hbar} = \frac{\omega_0'}{c}$ is finite as well, no need to renormalize ω_0'

Reduction of a wavicle to a QM particle

In many setting, QM works to a good approximation;
then wavicle reduction should be possible.

Recipe:

Construct $\hat{H}_{\rm QM}$ with spectrum of wavicle energies $E(k) = \hbar\omega(k)$.

Valid at low $|k|$, such that

$$\frac{2\pi}{|k|} \gg \lambda_\Phi = \frac{\hbar}{mc} \quad \Rightarrow \quad \hbar|k| \ll mc$$

Matches exactly non-relativistic limit for de Broglie momentum $\hbar|k|$,
if transition to quantum field theory converts c into speed of light.

Then QM is not that wrong: $\approx$ non-relativistic point particles

Example: H-atom, characteristic scale $r_{\rm Bohr} = \frac{\hbar^2}{e^2 m_e} \simeq 0.53\,\text{Å}$

QM for electron works quite well, because

$$\lambda_e = \frac{\hbar}{m_e c} = \alpha\, r_{\rm Bohr} \quad \text{with} \quad \alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036\ldots}$$

λ_e does not know about the electric charge. α relates it to the H-atom, at a larger scale, since the coupling is rather weak.
Still, high energy reveals corrections, in particular the Lamb shift.

Conclusion: Quantized field excitations ("wavicles", *i.e.* particles in a fundamental and covariant sense) are not point-like.
They decay exponentially over the range of the Compton wave length $\hbar/mc$; in the massless case (photons), they are non-local.

Thanks a lot to the LOC for the successful organization !

We acknowledge support by:

Facultad de Ciencias Físico Matemáticas, BUAP
Consejo Estatal de Ciencia y Tecnología del Estado de Puebla (CONCYTEP)

And we thank in particular:

Mario Rodriguez-Cahuantzi, Heber Zepeda,
Valeria Zelina Reyna Ortiz, Irandheny Yoval Pozos,
Javier Morales, Esther Huerta

and numerous helpful students of the *Cuerpo Acedémico de Partículas, Campos y Relatividad General de la FCFM-BUAP*