

# Fermion Spin Amplitudes: a Direct Computation Method

Matías Moreno <sup>\*</sup>, Rodolfo Leo <sup>\*</sup>,  
Genaro Toledo <sup>\*</sup> and Gabriel López-Castro <sup>\*\*,\*</sup>

<sup>\*</sup>*Instituto de Física, Universidad Nacional Autónoma de México, México, D.F. 04510, México*

<sup>\*\*</sup>*Departamento de Física, Centro de Investigación y Estudios Avanzados, México, D.F., México*

**Abstract.** The traditional method [1] to compute the square of transition amplitudes involving spin  $1/2$  particles is particularly difficult to use for spin polarized amplitudes. We present here an alternative method to calculate directly the Feynman amplitude through trace evaluation. This method is not constrained to a special type of polarization or particle mass.

**Keywords:** Spin amplitudes, Dirac algebra.

**PACS:** 13.88.+e, 13.90.+i

## STANDARD METHOD

The usual method to compute the transition probability between fermions  $i$  and  $f$  from the Feynman amplitude [1]

$$M = \bar{u}_f \Gamma u_i \quad (1)$$

is first to take its square

$$|M|^2 = \bar{u}_f \Gamma u_i \bar{u}_i \bar{\Gamma} u_f = \text{Tr}[P_f \Gamma P_i \bar{\Gamma}] \quad (2)$$

with  $P_x = \frac{1}{4m_x}(\not{p}_x + m_x)(1 + \gamma_5 \not{s}_x)$ .

Since  $|M|^2$  is a probability and *not* a probability *amplitude* this expression can be used to sum over polarizations. For a non polarized probability the the projector operator is  $P_x = \frac{1}{2m_x}(\not{p}_x + m_x)$ , which considerably simplifies the computation.

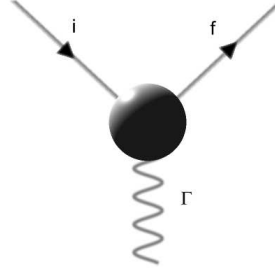
When one considers a physical process with two or more external fermion lines the generalization is immediate. If a single Feynman amplitude

$$M = \bar{u}_3 \Gamma_A u_1 \bar{u}_4 \Gamma_B u_2 \quad (3)$$

contributes, the computation proceeds for each fermionic line using the same procedure.

There are several limitations to this approach, the most important is the case in which many exchange amplitudes participate in the physical process. On the other hand, the most simple case of this nature is the electron-electron scattering. In this case there are two amplitudes that connect different fermions, for which the generic form is

$$\begin{aligned} M &= M_1 + M_2 \\ M_1 &= \bar{u}_3 \Gamma_A u_1 \bar{u}_4 \Gamma_B u_2 \\ M_2 &= \bar{u}_4 \Gamma_C u_1 \bar{u}_3 \Gamma_D u_2 \end{aligned}$$



**Figure 1.** Basic Amplitude.

If one squares this there are four terms to be computed

$$|M|^2 = |M_1|^2 + |M_2|^2 + (M_1 M_2^* + M_1^* M_2); \quad (4)$$

the trace expressions for the crossed term are difficult even in the zero polarization amplitude.

For amplitudes of more than two fermion lines (like the multiparton processes that appear in QCD), the situation only gets worse. This renders the analysis of the spin dependence cumbersome to say the least. Indeed

$$M_1 M_2^* = Tr[P_3 \Gamma_A P_1 \bar{\Gamma}_C P_4 \Gamma_B P_2 \bar{\Gamma}_D]. \quad (5)$$

which any reasonable person would prefer not to compute by hand.

The above discussion has been the origin of several attempts to produce the Feynman amplitudes directly without squaring them [2, 3]. The idea is to circumvent the crossed terms and to diminish the burden of computation, this last point may not be obvious for the passing observer but is in fact crucial. Indeed if one begins with  $N$  amplitudes taking the square in the first place renders  $N^2$  traces. Instead if the  $N$  amplitudes are computed first the only cost is that one cannot use the density matrix for the Dirac spinor projectors. All the direct amplitude calculation methods, to our knowledge, require at some point the selection of specific polarization directions for the fermion spin [4, 5]. In this work we present a method that does not suffer from this constraint.

## AMPLITUDE MODULUS CALCULATION

We now introduce a procedure that determines immediately the modulus of any non interchange amplitude. To illustrate it, let us consider the case of a single fermion line process, i.e. consider the amplitude in Eq. (1) and multiply it by unity in the following form

$$M = \bar{u}_f \Gamma u_i \times \frac{\bar{u}_i \Gamma_S u_f}{\bar{u}_i \Gamma_S u_f} \quad (6)$$

clearly for this to make sense, the factor  $\bar{u}_i \Gamma_S u_f$  must be *non null*. The  $\Gamma_S$  can be selected in the simplest possible way consistent with this requirement. Usually we can take  $\Gamma_S = 1$  or, in quiral conservation processes,  $\Gamma_S = (1 \pm \gamma^5)/2$ . The numerator of Eq. (6) is

$$\bar{u}_f \Gamma u_i \times \bar{u}_i \Gamma_S u_f = Tr[P_f \Gamma P_i \Gamma_S] \quad (7)$$

where  $P_x$  is a complete spinor projector (energy and polarization projectors) provided in the first section. This may not look simple but it is certainly less frightening than Eq. (5).

The denominator of Eq. (6) is the most simple non null Dirac amplitude. Its modulus is, as will be shown here, almost trivial to compute in a covariant way, the same cannot be said of its phase. Let us compute the modulus of

$$|\bar{u}_i \Gamma_S u_f|^2 = Tr[P_f \bar{\Gamma}_S P_i \Gamma_S] \quad (8)$$

Thus up to the phase  $e^{i\phi}$  of  $\bar{u}_i \Gamma_S u_f$  we have calculated

$$M = \bar{u}_f \Gamma u_i = \frac{Tr[P_f \Gamma P_i \Gamma_S]}{\sqrt{Tr[P_f \bar{\Gamma}_S P_i \Gamma_S]}} \quad (9)$$

which reduces to

$$M = \bar{u}_f \Gamma u_i = \frac{Tr[P_f \Gamma P_i]}{\sqrt{Tr[P_f P_i]}} \quad (10)$$

if  $\Gamma_S$  can be selected has 1.

## CONCLUSIONS

We have developed a (for us) new method to compute spin dependent Feynman amplitudes. The method relies on a simple (intelligent) multiplication by one. We presented here the computation of a non exchange amplitude, *i.e.* up to a phase factor. We have already evaluated this phase using either a Lorentz transformation or the Kleiss and Stirling method [2]. This as well as explicit applications will be presented elsewhere.

After finishing this work we became aware of the the work of A. L. Bondarev [6] that contains Eq. (6).

## ACKNOWLEDGMENTS

One of us (MM) would like to acknowledge the encouragement received from the Physics Department of Cinvestav during his sabbatical leave. There the early steps of this work were taken, he is especially indebted to Drs. Augusto García and Gerardo Herrera for their hospitality. This work is supported by Conacyt (México).

## REFERENCES

1. J. Bjorken and S. Drell, *Relativistic Quantum Mechanics*, McGraw Hill, (1966).
2. R. Kleiss and W. J. Stirling, *Nucl. Phys.* **B262**, 235 (1985), .
3. A. Hsieh and E. Yehudai, *Comput.Phys.* **6**, 253-261,(1992).
4. M. Veltman, *Nucl. Phys.* **B319**, 253 (1989) .
5. M. Moreno, *J. Math. Phys.* **26**,576 (1985).
6. A. L. Bondarev, hep-ph/9710398