ELECTROMAGNETIC MULTIPOLE MOMENTS OF SPIN 3/2 PARTICLES IN NKR FORMALISM

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Introduction

Recently, a new formalism for the description of particles with spin was proposed in [1]. The special case of spin 3/2 shows advantages with respect to the commonly used Rarita-Schwinger (RS) formalism. Unlike the RS method, in the formalism of Ref. [1] (NKR formalism here on) one gets a causal theory in the presence of an electromagnetic field.

In a previous work we calculated Compton Scattering off spin 3/2 particles in the NKR framework, we obtain the correct classical limit and the calculated cross section satisfy the unitarity constraints for center of mass energies up to the mass of the spin 3/2 particle. However, in the high energy limit, we obtain a cross section that violates unitarity. In this work we calculate the electromagnetic moments of a NKR particle.

NKR Formalism

In the NKR formalism, the equation of motion for a particle with spin s is the projection equation onto the eigensubspaces of the Casimir operators of the Poincaré group, the squared Pauli-Lubansky vector, W^2, and the squared four-momentum P^2.

$$\mathsf{P}^{(m;s)}(p)\Psi^{(m,s)}=\Psi^{(m,s)}.$$

In the simplest case of an irrep of the HLG containing only two Poincaré invariant subspaces whit spin $(s-1)$ and s, the mass m and spin s projector $\mathsf{P}^{(m;s)}(p)$ is

$$\mathsf{P}^{(m;s)}(p)=\frac{P^2}{m^2}\mathbb{P}^{(s)}=\frac{P^2}{m^2}\left[\frac{1}{2s}\left(-\frac{W^2}{m^2}-s(s-1)\frac{p^2}{m^2}\right)\right],$$

with $\quad [W_\lambda]_{AC}=\frac{1}{2}\varepsilon_{\lambda\rho\sigma\mu}\left[M^{\rho\sigma}\right]_{AC}p^\mu,$

$$\left[W^2\right]_{AB}=\frac{1}{4}\varepsilon_{\lambda\rho\sigma\mu}\left[M^{\rho\sigma}\right]_{AC}p^\mu\varepsilon^{\lambda}{}_{\tau\xi\nu}\left[M^{\tau\xi}\right]_{CB}p^\nu\equiv T_{AB\mu\nu}p^\mu p^\nu,$$

where $M_{\rho\sigma}$ are the generators of the HLG. If $\Gamma_{AB\mu\nu}=-[T_{AB\mu\nu}+s(s-1)\delta_{AB}g_{\mu\nu}]/(2s)$, the equation of motion can be written as

$$\left[\Gamma_{AB\mu\nu}p^\mu p^\nu-m^2g_{AB}\right]\Psi^B_{(m,s)}=0.$$

Spin 1 case

The simplest case in which this formalism can be applied is the spin 1 case, here one obtains the equation of motion as

$$\left[\Gamma_{\alpha\beta\mu\nu}p^\mu p^\nu-m^2g_{\alpha\beta}\right]V^\beta=0,$$

notice that we have the freedom to choose the antisymmetric part of the $\Gamma_{\alpha\beta\mu\nu}$ tensor, a general construction of it leads to

$$\Gamma_{\alpha\beta\mu\nu}=g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\mu}g_{\beta\nu}-ig\left[M^V_{\mu\nu}\right]_{\alpha\beta}+i\xi\varepsilon_{\alpha\beta\mu\nu},$$

here, the $M^V_{\mu\nu}$ are the generators of the $\left(\frac{1}{2},\frac{1}{2}\right)$ representation space. The Lagrangian associated with the equation of motion is

$$L=(\partial^\mu V^\alpha)\Gamma_{\alpha\beta\mu\nu}\partial^\nu V^\beta-m^2V^\alpha V_\alpha,$$

Spin 3/2 Case

Here we are interested in the $\left(\frac{1}{2},\frac{1}{2}\right)\otimes\left[\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right)\right]$ representation and the equation of motion is

$$\left[\tilde{\Gamma}_{\alpha\beta\mu\nu}p^\mu p^\nu-m^2g_{\alpha\beta}\right]\psi^\beta=0,\qquad\text{or}\qquad\left(p^2\mathbb{P}^{(3/2)}_{\alpha\beta}-m^2g_{\alpha\beta}\right)\psi^\beta=0,$$

where

$$\tilde{\Gamma}_{\alpha\beta\mu\nu}=\tfrac{1}{6}(\varepsilon^{\lambda}{}_{\alpha\beta\mu}\gamma^5\sigma_{\lambda\nu}+\varepsilon^{\lambda}{}_{\alpha\beta\nu}\gamma^5\sigma_{\lambda\mu})+\tfrac{1}{12}\sigma_{\lambda\nu}\sigma^{\lambda}{}_{\nu}g_{\alpha\beta}-\tfrac{1}{4}g_{\mu\nu}g_{\alpha\beta}+\tfrac{2}{3}(g_{\alpha\beta}g_{\mu\nu}-g_{\alpha\nu}g_{\beta\mu}),$$

$$\mathbb{P}^{(3/2)}_{\alpha\beta}=-\frac{1}{3}\left(\frac{W^2{}_{\alpha\beta}}{p^2}+\frac{3}{4}g_{\alpha\beta}\right).$$

This equation implies the following constrictions:

$$\left(p^2-m^2\right)\psi_\alpha=0,\qquad\gamma^\alpha\psi_\alpha=0,\qquad p^\alpha\psi_\alpha=0.$$

The propagator can be found as the inverse of the $\tilde{\Gamma}_{\alpha\beta\mu\nu}p^\mu p^\nu-m^2g_{\alpha\beta}$ operator:

$$\Pi_{\alpha\beta}(p)=\frac{\mathbb{P}^{(3/2)}_{\alpha\beta}}{p^2-m^2}-\frac{\mathbb{P}^{(1/2)}_{\alpha\beta}}{m^2}=\frac{\mathbb{P}^{(3/2)}_{\alpha\beta}-\frac{p^2-m^2}{m^2}\mathbb{P}^{(1/2)}_{\alpha\beta}}{p^2-m^2}\equiv\frac{\Delta_{\alpha\beta}(p)}{p^2-m^2},$$

where the spin 1/2 projector in ψ_μ is

$$\mathbb{P}^{(1/2)}_{\alpha\beta}=g_{\alpha\beta}-\mathbb{P}^{(3/2)}_{\alpha\beta}=\frac{W^2{}_{\alpha\beta}}{3p^2}+\frac{5}{4}g_{\alpha\beta}.$$

Spin 3/2 Lagrangian

Clearly, the projector equation fixes the symmetric part of $\tilde{\Gamma}_{\alpha\beta\mu\nu}$ under the exchange of $\mu\leftrightarrow\nu$ but the anti-symmetric part $\tilde{\Gamma}^A_{\alpha\beta\mu\nu}$ is arbitrary. The most general form for the antisymmetric part is constructed from Lorentz invariants and the involved free parameters were fixed requiring hermiticity and the suppression of the 3/2-1/2 transitions in the presence of electromagnetic interactions introduced via the gauge principle $\partial^\alpha\rightarrow D^\alpha=\partial^\alpha-ieA^\alpha$. Under these constraints, the antisymmetric part depends only on two parameters, f and g, which fixes the magnetic moment and higher multipoles. Studying the propagation of the spin 3/2 waves in an electromagnetic background , it is shown in [1] that the only values of g leading to causal propagation are $g=0,2$. The latter value is chosen as appropriate for the description of charged particles and a striking connection between causality and the gyromagnetic ratio $g=2$ is established this way. In terms of this two parameters we have

$$\Gamma_{\alpha\beta\mu\nu}=\tfrac{1}{3}(i\sigma_{\alpha\beta}+2g_{\alpha\beta})g_{\mu\nu}+\tfrac{1}{3}(i\sigma_{\beta\nu}-2g_{\beta\nu})g_{\alpha\mu}-\tfrac{1}{3}i\sigma_{\alpha\mu}g_{\beta\nu}-ig\left[M_{\mu\nu}\right]_{\alpha\beta}-if\gamma^5\varepsilon_{\alpha\beta\mu\nu}$$

The Lagrangian density associated with the equation of motion is

$$L=(\partial^\mu\bar{\psi}^\alpha)\Gamma_{\alpha\beta\mu\nu}\partial^\nu\psi^\beta-m^2\bar{\psi}^\alpha\psi_\alpha.$$

Gauging this Lagrangian we get the interacting part as

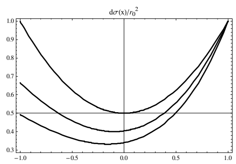
$$L_{\rm int}=j_\mu A^\mu+e^2\bar{\psi}^\alpha\Gamma_{\alpha\beta\mu\nu}\psi^\beta A^\mu A^\nu.$$

Using $\psi_\alpha=u_\alpha(p)e^{-ip\cdot x}$, we find the transition current in momentum space as

$$j_\mu=e\bar{u}^\alpha(p')(\Gamma_{\alpha\beta\nu\mu}p'^\nu+\Gamma_{\alpha\beta\mu\nu}p^\nu)u^\beta(p)=\bar{u}^\alpha(p')O(p',p)_{\alpha\beta\mu}u^\beta(p).$$

Compton Scattering

This is the simplest process in which an electromagnetic field interacts with a charged particle. At the tree level, a particle with zero spin interacts only with its charge, unlike a particle with spin 1/2 that interacts also with its magnetic dipole moment. Using the Dirac's equation of motion one can find the form of the interaction and the Feynman rules involved. The angular distribution of the process is a known result:



Angular distribution for the Compton scattering off spin 1/2 particles in the lab frame. The symmetric curve corresponds to the classical limit η=0, the other curves correspond to η=0.12 and η=0.3

One can use the Compton Scattering process to get more information about the undetermined parameters appearing in the interaction currents obtained for spin 1 and spin 3/2 above. However, in the spin 3/2 case, we do not consider odd parity contributions for simplicity. In the classical limit, the cross section in the lab frame for this process is independent of undetermined parameters:

$$\sigma_{s=1}(g,\xi)\big|_{\eta\to0}=\frac{8\pi}{3}r_0^2\equiv\sigma_T,\qquad\sigma_{s=3/2}(f,g)\big|_{\eta\to0}=\frac{8\pi}{3}r_0^2\equiv\sigma_T$$

Where $\eta=\omega/m$, ω being the energy of the incident photon, $r_0=\alpha/m$ is the so called classical radius of the particle and σ_T the Thompson cross section. This result is what one expects since at low energies the particle interacts only with its lower moment, the charge. Higher moments become relevant with increasing energy.

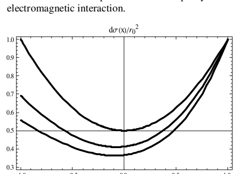
In the case of spin 1, at very high energies one gets a divergent cross section for arbitrary parameters g and ξ, in order to determine whether there are any values that preserves unitarity, one has to integrate the differential cross section from $x=-1+\epsilon$ to $x=1-\epsilon$, where $x=\cos\theta_{\rm lab}$. In this limit we find [2]:

$$\begin{aligned}\sigma_{s=1}(g,\xi)\big|_{\eta\to\infty}=\frac{8\pi r_0^2}{3}\Bigg[&\frac{1}{128}2(1-\varepsilon)(g^2+4g-4+\xi^2)^2\\&+2\left((g-2)^2+\xi^2\right)(7g^2-12g+12+7\xi^2)\left(\frac{1}{\varepsilon-2}+\frac{1}{\varepsilon}\right)\\&+2\left((g-2)^2+\xi^2\right)(3g^2+8g+-4+3\xi^2)Log\left(\frac{1}{\varepsilon}-1\right)\Bigg],\end{aligned}$$

Looking at this expression one can conclude that the only values that preserve unitarity at high energies when ϵ goes to zero are

$$g=2,\qquad\xi=0.$$

This is a result that agrees whit the usual value of the gyromagnetic ratio g, we also confirm our assumption that odd parity terms must not contribute to the electromagnetic interaction.

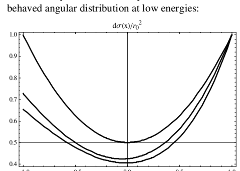


Angular distribution for the Compton scattering off spin 1 particles in NKR formalism in the lab frame. The symmetric curve corresponds to the classical limit η=0, the other curves correspond to η=0.12 and η=0.3

Now we are interested in the high energy limit of the Compton Scattering cross section in the spin 3/2 case. Here we obtain also a divergent cross section for arbitrary parameters f and g, but applying the same procedure as before we find that the partially integrated cross section is

$$\sigma_{s=3/2}(f,g)\big|_{\eta\to\infty}=\frac{\mathrm{sign}(f^2+fg+g^2)^2\,\mathrm{sign}(\varepsilon-1)}{\mathrm{sign}(\varepsilon-2)}\times\infty$$

(result given by the Feyncalc package) This expression means that still with this procedure we find that independently of the choice of parameters, the cross section will diverge. At low energies, we get a good behavior of the cross section, in the region dominated by the lowest multipole moments. If we make $g=2$ and $f=0$ we obtain a well behaved angular distribution at low energies:



Angular distribution for the Compton scattering off spin 3/2 particles in NKR formalism in the lab frame. The symmetric curve corresponds to the classical limit η=0, the other curves correspond to η=0.12 and η=0.3

In the region of high energy, the cross section must be dominated by the higher moments. The main objective of this work is to calculate these electromagnetic multipole moments, aiming to understand the high energy behavior of the Compton cross section.

Electromagnetic moments of vector particles in NKR Formalism

The NKR Lagrangian for vector particles in the presence of an electromagnetic field is

$$L=(D^\mu V^\alpha)\Gamma_{\alpha\beta\mu\nu}D^\nu V^\beta-m^2V^\alpha V_\alpha,$$

where $D^\alpha=\partial^\alpha-ieA^\alpha$ .The interaction Lagrangian is

$$L_{\rm int}=j_\mu A^\mu+e^2V^\alpha\Gamma_{\alpha\beta\mu\nu}V^\beta A^\mu A^\nu.$$

In momentum space the current can be written as

$$j_\mu=e\eta^{*\alpha}(p')(\Gamma_{\alpha\beta\nu\mu}p'^\nu+\Gamma_{\alpha\beta\mu\nu}p^\nu)\eta^\beta(p).$$

We can make a Gordon decomposition of the current:

$$j_\mu(p',p)=e\eta^{*\alpha}(p',\lambda')\left(g_{\alpha\beta}(p'+p)_\mu-ig\left[M^V_{\mu\nu}\right]_{\alpha\beta}(p'-p)^\nu+\xi\varepsilon_{\alpha\beta\mu\nu}(p'-p)^\nu\right)\eta^\beta(p,\lambda).$$

Here we can identify g as the gyromagnetic factor, since a current of this form has a dipole magnetic moment operator related to the spin operator by the relation

$$\hat{\mu}=-\frac{ge}{2m}\hat{S},$$

so that between states of same polarization and with a magnetic field aligned in the z-direction we have

$$\langle\lambda|\hat{\mu}|\lambda\rangle=-\lambda\frac{ge}{2m}.$$

This value and the expectation values for all nonzero multipole moments at this order can also be obtained from this current by its classical definition [3], for example, the charge is:

$$E1=\lim_{\vec{q}\to0}\rho(\vec{q})=-e$$

And for higher moments we use the definitions [3]:

$$E2=\lim_{\vec{q}\to0}\left(-i\frac{\partial}{\partial q_3}\rho(\vec{q})\right),\qquad E4=\lim_{\vec{q}\to0}\left(\left(\frac{\partial^2}{\partial q_1^2}+\frac{\partial^2}{\partial q_2^2}-2\frac{\partial^2}{\partial q_3^2}\right)\rho(\vec{q})\right)$$

$$M2=\lim_{\vec{q}\to0}\left(\frac{i}{2}\frac{\partial}{\partial q_3}\left(\frac{\partial}{\partial\vec{q}}\cdot(\vec{j}\times\vec{q})\right)\right),\quad M4=\lim_{\vec{q}\to0}\left(\frac{1}{3}\left(\frac{\partial^2}{\partial q_1^2}+\frac{\partial^2}{\partial q_2^2}-2\frac{\partial^2}{\partial q_3^2}\right)\frac{\partial}{\partial\vec{q}}\cdot(\vec{j}\times\vec{q})\right)$$

Where $E2$ ($M2$) is the electric (magnetic) dipole moment and $E4$ ($M4$) is the electric (magnetic) quadrupole moment. The charge and current densities are defined in terms of the interaction current between states in the Breit frame:

$$\rho(\vec{q})=j^0(p',p),\qquad j^i(\vec{q})=j^i(p',p),\qquad p'=(\omega/2,\vec{q}/2),\qquad p=(\omega/2,-\vec{q}/2)$$

The results obtained for the multipole moments of a vector particle between states of polarization λ are

$$E2=\lambda\frac{e\xi}{2m},\qquad M2=\lambda\frac{eg}{2m},$$

$$E4=(3\lambda^2-2)\frac{e(g-1)}{m^2},\qquad M4=(3\lambda^2-2)\frac{e\xi}{m^2},$$

reproducing the expected result for the dipole magnetic moment. Higher moments are zero. The values of g and ξ can be fixed to 2 and zero respectively by the requirement of unitarity of Compton scattering in the high energy limit [2]. These values reproduce the structure of the W^+ and is consistent with the fact that the contribution of the ξ term, which gives rise to electric dipole and a magnetic quadrupole moments, is parity odd.

Electromagnetic moments of spin 3/2 particles in the NKR Formalism.

The Gordon decomposition of the transition current for spin 3/2 is:

$$j_\mu(p',p)=e\bar{u}^\alpha(p',\lambda')\left(g_{\alpha\beta}(p'+p)_\mu-ig\left[M^{3/2}_{\mu\nu}\right]_{\alpha\beta}(p'-p)^\nu+if\gamma^5\varepsilon_{\alpha\beta\mu\nu}(p'-p)^\nu\right)u^\beta(p,\lambda).$$

Again, one can identify g as the gyromagnetic factor, since it can be demonstrated that

$$\hat{\mu}=-\frac{ge}{2m}\hat{S},$$

In this case we expect our current to include higher multipole moments. We calculate all the multipole moments using the definitions above and [3]:

$$E8=\lim_{\vec{q}\to0}\left(-3i\frac{\partial}{\partial q_3}\left(3\frac{\partial^2}{\partial q_1^2}+3\frac{\partial^2}{\partial q_2^2}-2\frac{\partial^2}{\partial q_3^2}\right)\rho(\vec{q})\right)$$

$$M8=\lim_{\vec{q}\to0}\left(\frac{-3i}{4}\frac{\partial}{\partial q_3}\left(3\frac{\partial^2}{\partial q_1^2}+3\frac{\partial^2}{\partial q_2^2}-2\frac{\partial^2}{\partial q_3^2}\right)\frac{\partial}{\partial\vec{q}}\cdot(\vec{j}\times\vec{q})\right)$$

Where $E8$ ($M8$) is the electric (magnetic) octupole moment. In this case, considering even parity contributions only, the multipole moments of a spin 3/2 particle between states of polarization $\lambda=3/2,1/2,-1/2,-3/2$ are

$$E2=0,\qquad M2=\lambda\frac{eg}{2m},$$

$$M4=0,\qquad E4=\frac{1}{3}\left(3\lambda^2-\frac{15}{4}\right)\frac{e(-f+g-1)}{m^2},$$

$$E8=0,\qquad M8=\lambda\left[\frac{15}{2}\lambda^2-\frac{123}{8}\right]\frac{e(2f+g)}{m^3}$$

reproducing the expected result for the dipole magnetic moment.

Conclusions

The new results in this work are the electromagnetic multipole moments of a spin 3/2 particle in NKR formalism. We have found that this formulation produces an electromagnetic interaction that includes the expected number of nonzero multipole moments of a spin 3/2 particle. In this formalism, apart from the charge, particles interact with the electromagnetic field via its magnetic dipole moment, electric quadrupole moment, and, in addition with respect to the vector case, via its octupole magnetic moment.

We found that the term in the current proportional to the generators

$$e\bar{u}^\alpha(p',\lambda')\left(ig\left[M^{3/2}_{\mu\nu}\right]_{\alpha\beta}(p'-p)^\nu\right)u^\beta(p,\lambda).$$

contain contributions to all the relevant multipole moments. We have also found that the new term coming from a general formulation

$$e\bar{u}^\alpha(p',\lambda')\left(if\gamma^5\varepsilon_{\alpha\beta\mu\nu}(p'-p)^\nu\right)u^\beta(p,\lambda).$$

can contribute to the electric quadrupole moment and the magnetic octupole moment of the particle.

The next step in the Compton scattering problem is to study the origin of violation of unitarity of the scattering of a spin 3/2 particle on the light of the present results.

References

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