Gauge Transformations as Spacetime Symmetries

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Abstract. Weinberg has shown that massless fields of helicity ± 1 (vector fields) do not transform homogeneously under Unitary Lorentz Transformations (LT). We calculate explicitly the inhomogeneous term. We show that imposing strict invariance of the Lagrangian under LT for an iteracting Dirac field requires the fermion field to transform with a space-time (and photon creation and annihilation operators) dependent phase and dictates the interaction terms as those arising from the conventional gauge principle.

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A unitary representation of the Poincaré group associates to each Poincaré transformation b^μ, Λ_1 (where b^μ is a space time translation and Λ_1 an homogeneous Lorentz transformation) a unitary operator $U(b,\Lambda_1)$, satisfying the composition rule $U(b,\Lambda_1)U(c,\Lambda_2) = U(b+\Lambda_1 c, \Lambda_2\Lambda_1)$. The transformation properties of free particle states under the Poincaré group is essentially included in the corresponding little group. For massless particles this subgroup is given by the subset of Lorentz Transformations (LT) leaving invariant light-like four vectors, i.e. given k^μ light-like, $\mathscr{R}$ belongs to the little group iff $\mathscr{R}k^\mu = k^\mu$. For the sake of simplicity we take a representative of light-like vector as $k^\mu = (k,0,0,k)$, $k>0$. A massless free particle with helicity σ and momentum $\mathbf{p}$ is described by the state $|\Psi_{p,\sigma}\rangle$ transforming under unitary Lorentz transformation, $U(0,\Lambda) = U(\Lambda)$, as [2]

$$U(\Lambda)|\Psi_{p,\sigma}\rangle = \sqrt{\frac{(\Lambda p)^0}{p^0}} exp(i\sigma\theta(\mathscr{R}(\Lambda,p)))|\Psi_{\Lambda p,\sigma}\rangle, \tag{1}$$

where θ is the Wigner angle [1].

The physical requirement of *locality* forces us to introduce creation and annihilation operators for the particle states, $|\Psi_{p,\sigma}\rangle = a^\dagger(\mathbf{p},\sigma)|0\rangle$. The transformation rule for the creation and annihilation operators of the states $|\Psi_{p,\sigma}\rangle$, is dictated by Eq. (1) as

$$U(\Lambda)a^\dagger(\mathbf{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}} exp(i\sigma\theta(\Lambda,p)), a^\dagger(\mathbf{p}_\Lambda,\sigma). \tag{2}$$

Our physical theories must also be *causal*, thus we must work in configuration space leading us to the integration of creation and annihilation operators over all spatial momenta $\mathbf{p}$. This procedure configures a field which in order to produce a Lorentz covariant S matrix must transform in the irreducible representations (non-necessarily unitary thus finite-dimensional) of the Lorentz Group which is isomorphic to $SU(2)_A \otimes SU(2)_B$ where $\mathbf{A} = \mathbf{J} + \mathbf{iK}$, $\mathbf{B} = \mathbf{J} - \mathbf{iK}$ and $\mathbf{J}$, $\mathbf{K}$ denote the rotation and boost generators. These irreps are labelled as (a,b). A field $\phi^{ab}(x)$ of massless neutral particles is constructed as linear combination of annihilation and creation operators

$$\phi^{ab}(x) = \sum_{\{\sigma\}} \int \frac{d^3p}{\sqrt{(2\pi)^3(2p_0)}} \left\{ e^{ip\cdot x} u^{ab}(\mathbf{p},\sigma) a(\mathbf{p},\sigma) + e^{-ip\cdot x} v^{ab}(\mathbf{p},\sigma) a^\dagger(\mathbf{p},\sigma) \right\}, \quad (3)$$

where $\{\sigma\}$ denotes the possible particle helicities. Weinberg [2] has shown that a massless field of type (a,b) can be formed only from annihilation operators with helicity σ, and creation operators with helicity $-\sigma$, where $\sigma = b - a$. In particular a photon field must have helicities ± 1 thus we are only allowed to use fields of the type $(n, n-1) \oplus (n-1, n)$ for the photon. Weinberg [3] has also shown that in a Lorentz invariant and perturbative S-matrix theory, these fields give amplitudes that vanish for small photon momentum $p \to 0$ in contradiction with experiment. The classical photon field uses the $(\frac{1}{2}, \frac{1}{2})$ representation. However, Weinberg proved that is impossible to construct a helicity ± 1 massless vectorial field that transforms homogeneously under LT [2]. The explicit calculation of the inhomogeneous term and its conexion with gauge transformations is the aim of this work. For the $(\frac{1}{2}, \frac{1}{2})$ representation the field can be recast in terms of Lorentz indices

$$A^\mu(x) = \sum_{\{\sigma\}} \int \frac{d^3p}{(2\pi)^{3/2}\sqrt{(2p_0)}} \left\{ e^{ip\cdot x} u^\mu(\mathbf{p},\sigma) a(\mathbf{p},\sigma) + e^{-ip\cdot x} v^\mu(\mathbf{p},\sigma) a^\dagger(\mathbf{p},\sigma) \right\}. \quad (4)$$

Using Eq. (2) we obtain the transformation properties of this field as

$$\begin{aligned} U(\Lambda)A^\mu(x)U^{-1}(\Lambda) &= \int \frac{d^3(\Lambda p)}{(2\pi)^{3/2}\sqrt{2(\Lambda p)^0}} \{ e^{ip\cdot x} u^\mu(\mathbf{p},\sigma) exp(-i\sigma\theta(\Lambda,p)) a(\mathbf{p}_\Lambda,\sigma) \\ &+ e^{-ip\cdot x} v^\mu(\mathbf{p},\sigma) exp(i\sigma\theta(\Lambda,p)) a^\dagger(\mathbf{p}_\Lambda,\sigma) \}. \end{aligned} \quad (5)$$

Weinberg [3] has shown that the coefficients transform as

$$\left[(\Lambda^{(-1)})^\mu_{\ \nu} - \Lambda_\nu^{\ 0} \frac{p^\mu}{p^0} \right] u^\nu(\mathbf{p}_\Lambda,\sigma) = exp(-i\sigma\theta(\mathbf{p},\Lambda)) u^\mu(\mathbf{p},\sigma). \quad (6)$$

the coefficient v^μ transforming as the complex conjugate of this equation. Using these transformation rules in Eq.(5) we obtain

$$U(\Lambda)A^\mu(x)U^{-1}(\Lambda) = (\Lambda^{(-1)})^\mu_{\ \nu} A^\nu(\Lambda x) + i\frac{\partial}{\partial(\Lambda x)_\mu}\Omega(\Lambda,x), \quad (7)$$

where $\Omega(\Lambda,x)$ is a function of space-time, linear in the photon annihilation and creation operators given by

$$\Omega(\Lambda,x) = \int \frac{d^3(p)}{\sqrt{(2\pi)^3 2p^0}} \frac{(\Lambda^{(-1)})^0_{\ \nu}}{(\Lambda^{-1}p)^0} \left\{ e^{ip\cdot\Lambda x} u^\nu(\mathbf{p},\sigma) a(\mathbf{p},\sigma) - e^{-ip\cdot\Lambda x} u^{*\nu}(\mathbf{p},\sigma) a^\dagger(\mathbf{p},\sigma) \right\}. \quad (8)$$

Under a LT $\Omega(\Lambda,x)$ transforms as

$$U(\Lambda_2)\Omega(\Lambda_1,x)U(\Lambda_2)^{-1} = \Omega(\Lambda_2\Lambda_1,x) - \Omega(\Lambda_2,\Lambda_1 x), \tag{9}$$

thus it not a scalar field. In order to exhibit the effect of this term for interacting theories let us consider electromagnetic interactions of a Dirac field. The interacting term cannot be constructed with derivatives of the field A_μ as discussed above thus the most general lagrangian is

$$\mathscr{L}(x) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi + J_\mu A^\mu. \tag{10}$$

where J_μ denotes the (unknown) fermion current. With the conventional transformation rules for fermions, under a LT this Lagrangian transforms as

$$U(\Lambda)\mathscr{L}(x)U^{-1}(\Lambda) = \mathscr{L}(\Lambda x) - i\widetilde{J}_\mu\frac{\partial}{\partial(\Lambda x)_\mu}\Omega(\Lambda,x) \tag{11}$$

The last term can be integrated by parts and using current conservation we get Lorentz invariance up to a surface term related to $\Omega(\Lambda,x)$ which is harmless in perturbative calculations. However this term *must* be considered for strongly coupled theories like QCD. The cancellation of this term can be achieved if the Dirac field transforms under LT with a phase related to $\Omega(\Lambda,x)$

$$U(\Lambda)\Psi^l U^{-1}(\Lambda) = e^{-i\Omega(\Lambda,x)}(e^{i/2w(\Lambda)_{\mu\nu}\mathscr{J}^{\mu\nu}})^l_m\Psi^m(\Lambda x) \tag{12}$$

For two unitary LTs Λ_1 and Λ_2 acting on the Dirac field we get:

$$U(\Lambda_2)U(\Lambda_1)\Psi^l U^{-1}(\Lambda_1)U^{-1}(\Lambda_2) = e^{-i\Omega(\Lambda_2\Lambda_1,x)}(e^{i/2w(\Lambda_2\Lambda_1)_{\mu\nu}\mathscr{J}^{\mu\nu}})^l_m\Psi^m(\Lambda_2\Lambda_1 x), \tag{13}$$

thus Eq.(12) is indeed a representation of the Lorentz group. It can be easily proven that with this transformation rule for the Dirac field the lagrangian is strictly invariant whenever $J_\mu = q\ \bar{\Psi}\gamma_\mu\Psi$, i.e. strict Lorentz invariance of the theory dictates the very same interacting terms as those obtained using the gauge principle.

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