Perturbative and Non-Perturbative Phenomena in QED:

The Scattering by a Solenoidal Magnetic Field

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- For Coulomb scattering, the differential cross section in lowest order in perturbation theory, equals the classical Rutherford's result.

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  Aharonov and Bohm, *Phys. Rev.* **115**, 485 (1959)

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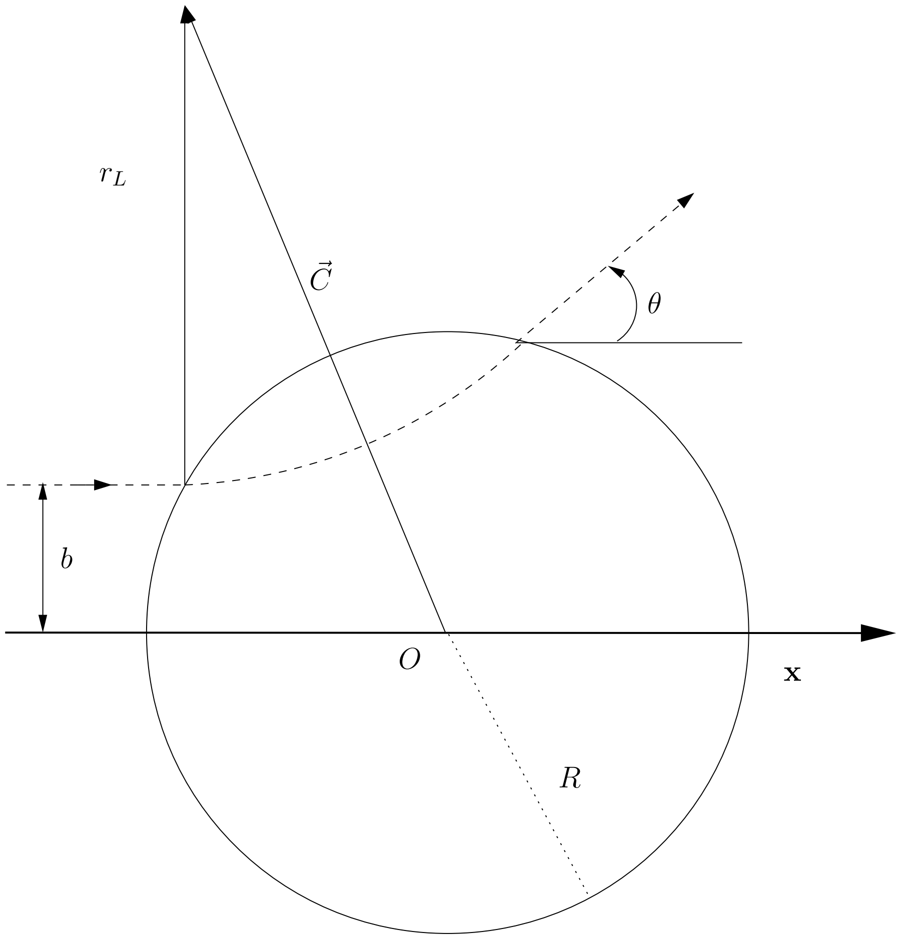
- Why it is important?

  Magnetic fields confine charged particles (pQCD?)

Classical regime



Solenoid of radius R

Magnetic field: $\mathbf{B} = B_0\mathbf{z}$

Magnetic flux (constant):

$\Phi = \pi R^2 B_0$

Impact parameter:

$\rho_b = b/R \in [-1, 1]$

Larmor radius:

$r_L = \frac{pc}{eB}$, $\rho_L \equiv r_L/R$

Scattering angle: $\theta \in [0, 2\pi)$

Non-radiation assumption

Classical regime 2

From classical mechanics:

$$\theta(\rho_b) = 2\arctan\left(\frac{\sqrt{1-\rho_b^2}}{\rho_b+\rho_L}\right)$$

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$$\rho_b^{\pm}(\theta, \rho_L) \;=\; -\rho_L \sin^2(\theta/2) \pm \cos(\theta/2)\sqrt{1-\rho_L^2 \sin^2(\theta/2)}$$

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$$\rho_b(\theta,\rho_L<1) = \rho_b^{+}(\theta,\rho_L) \text{ for } \theta\in[0,2\pi)$$

$$\rho_b(\theta,\rho_L\geq 1) = \rho_b^{\pm}(\theta,\rho_L) \text{ for } \theta\in[0,\theta_{\text{max}}],$$
$$\sin(\theta_{\text{max}}/2) = 1/\rho_L$$

Classical regime 3

The differential cross section: $\frac{d\sigma(\theta)}{d\theta} = \sum_i \left|\frac{db_i(\theta)}{d\theta}\right|$

$$\frac{1}{R}\frac{d\sigma(\theta)}{d\theta} = \left|\frac{\sin\theta}{2}\left(\rho_L + \frac{1+\rho_L^2\cos\theta}{2\cos\left(\theta/2\right)\sqrt{1-\rho_L^2\sin^2\left(\theta/2\right)}}\right)\right|$$

$$+\left|\frac{\sin\theta}{2}\left(\rho_L - \frac{1+\rho_L^2\cos\theta}{2\cos\left(\theta/2\right)\sqrt{1-\rho_L^2\sin^2\left(\theta/2\right)}}\right)\right|\Theta(|\rho_L|-1)$$

$\theta \in [0, 2\pi)$ if $\rho_L < 1$ and $\theta \in [0, \theta_{\text{max}}]$ if $\rho_L \geq 1$.

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$$+ \left| \frac{\sin\theta}{2} \left(\rho_L - \frac{1+\rho_L^2\cos\theta}{2\cos(\theta/2)\sqrt{1-\rho_L^2\sin^2(\theta/2)}} \right) \right| \Theta(|\rho_L|-1)$$

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$$\rho_L \rightarrow -\rho_L \equiv \theta \rightarrow 2\pi - \theta$$

Classical regime 4

Action-dimension parameters:

$$\hbar s_p = pR$$

$$\hbar s_\Phi = \frac{e\Phi}{c}$$

Relevant classical parameter:

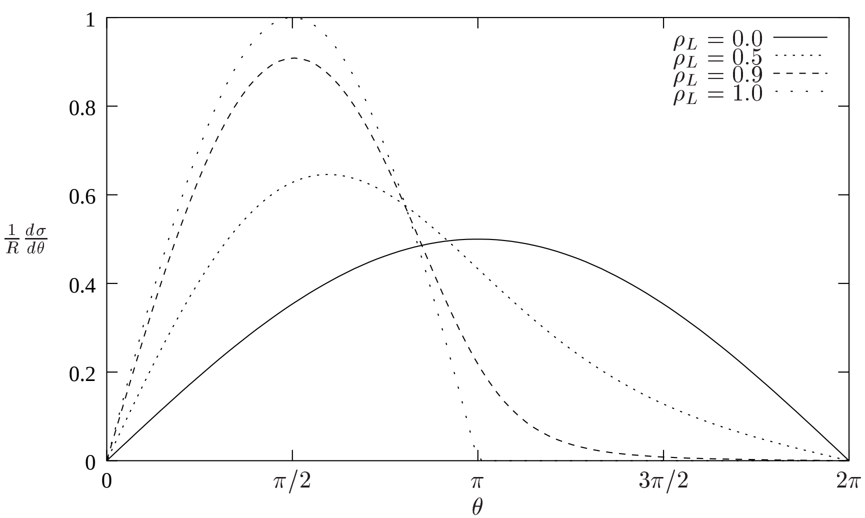
$$\rho_L = \frac{r_L}{R} = \pi \frac{s_p}{s_\Phi}$$

Classical regime 5

Rigid solenoid case

$$\rho_L \ll 1$$

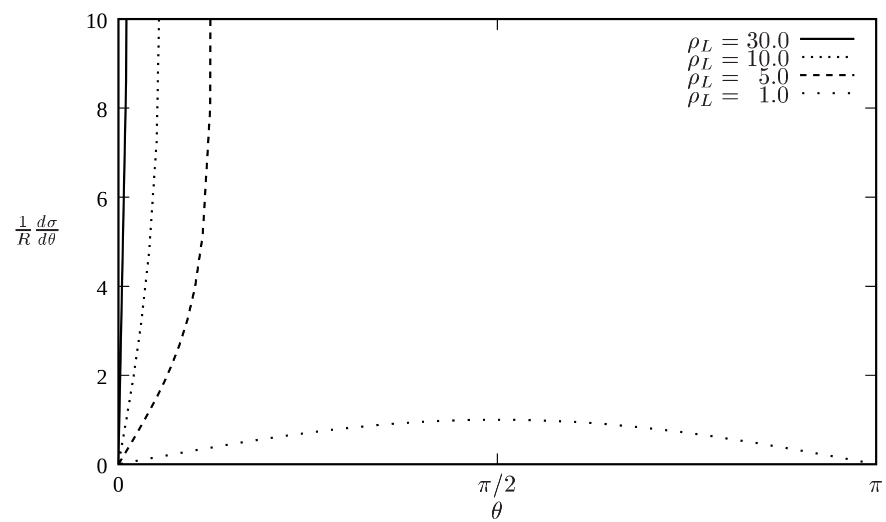
$$\left.\frac{d\sigma}{d\theta}\right|_{\rho_L \to 0} = \frac{R}{2}\,|\sin(\theta/2)|$$



Weak field limit

$$\rho_L \gg 1$$

$$\left.\frac{d\sigma}{d\theta}\right|_{s_\Phi \ll 1} \approx 2\pi R \frac{s_p}{s_\Phi} \frac{u}{\sqrt{1-u^2}}$$



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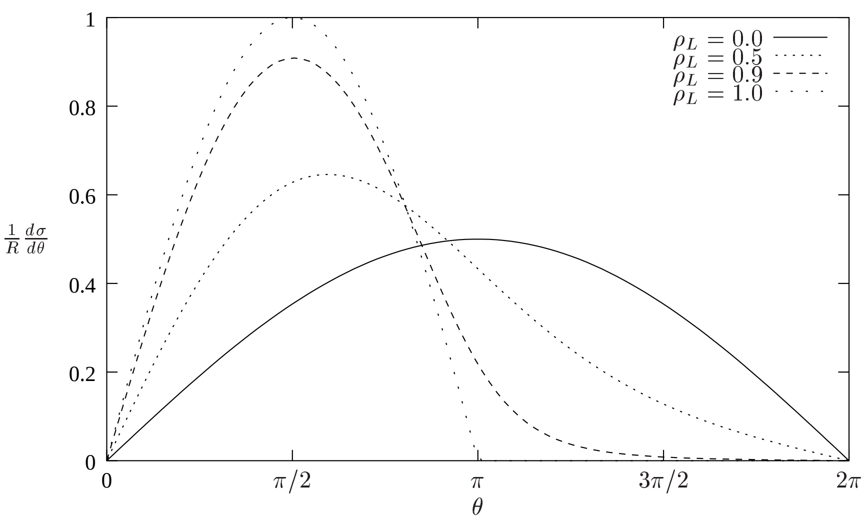

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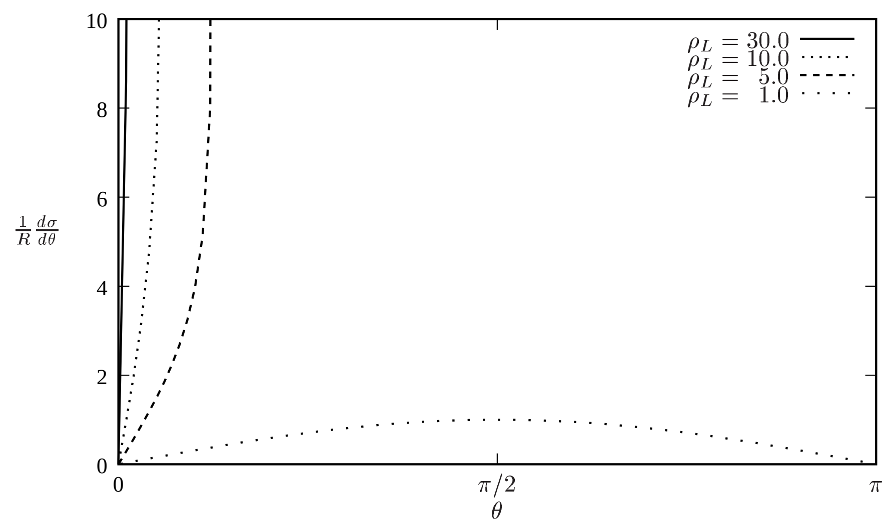
Perturbative analysis implies:

$\hbar s_\Phi \ll 1 \equiv \rho_L \gg 1$ $(s_\Phi = e\Phi/\hbar c)$

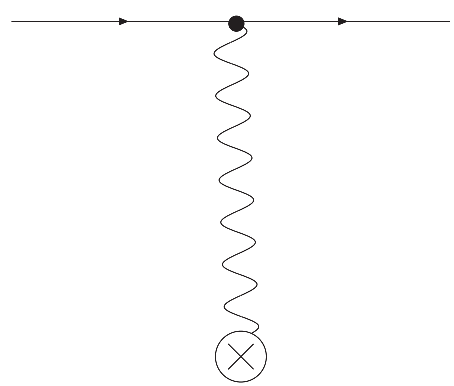


Q-Perturbative Analysis

Relativistic quantum perturbative analysis

Murguia and Moreno, *J. Phys.* **A36**, 2545 (2003)

- Free particle asymptotic sates
- Constant magnetic flux: $\Phi = \pi R^2 B_0$



$$S_{fi}^{(1)} = \delta_{fi} - i \int \bar{\psi}_f(x) \frac{e\not{A}(x)}{\hbar c} \psi_i(x) d^4x$$

Q-Perturbative analysis 2

Free particle solutions:

$$\psi(x) = \sqrt{\frac{mc^2}{EV}} u(\mathbf{p},\mathbf{s}) e^{-ip\cdot x/\hbar}$$

Magnetic potential of the solenoid:

$$\not{A} = A_\mu \gamma^\mu = \frac{\Phi}{2\pi} \epsilon_{ij3} x_i \gamma^j \begin{cases} \frac{1}{R^2} & \text{for } r < R \\ \frac{1}{x_1^2 + x_2^2} & \text{for } r > R \end{cases}$$

Q-Perturbative analysis 3

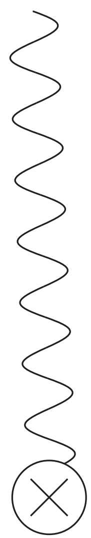
Feynman rules:

$$q^\mu = q^\mu_\parallel + q^\mu_\perp \qquad q^\mu_\parallel = (q_0, 0, 0, q_3), q^\mu_\perp = (0, q_1, q_2, 0)$$

$u(\mathbf{p}, \mathbf{s})$

$\bar{u}(\mathbf{p}, \mathbf{s})$

$e\Phi/\hbar c$



$$A(q) = -2i\frac{\hbar^2}{R} J_1(q_\perp R/\hbar)\epsilon_{ij3}\frac{q_i}{q_\perp^3}\gamma^j$$

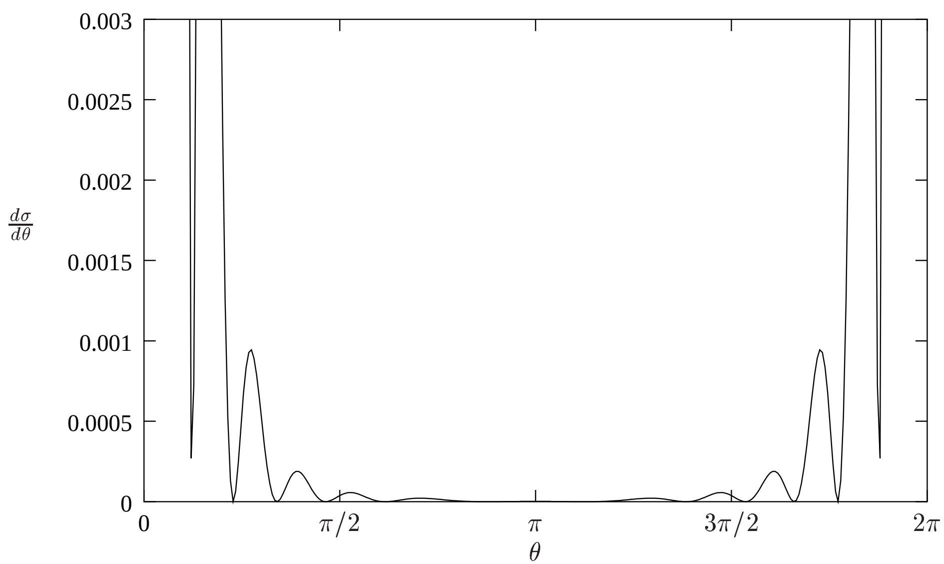
$$-iS_F(q) = -i\,\frac{\hbar}{q^\mu\gamma_\mu - mc + i\epsilon}$$

Q-Perturbative analysis 4

Differential cross section to first order
in $\alpha = e^2/\hbar c$ and $\beta = e\Phi/2\pi c$:

$$\frac{d\sigma}{d\theta} = \hbar \left(\frac{e\Phi}{Rc} \right)^2 \frac{\left| J_1(2\frac{p}{\hbar} R|\sin{(\theta/2)}|) \right|^2}{8\pi p^3 \sin^4{(\theta/2)}}$$

$$d\sigma(2\pi - \theta) = d\sigma(\theta) \Longrightarrow \text{Asimmetry} \equiv 0$$



Q-Perturbative analysis 5

The result is consistent:

We recover the Aharonov-Bohm result for $\frac{e\Phi}{2\hbar c} \ll 1$:

$$\left.\frac{d\sigma}{d\theta}\right|_{\frac{p}{\hbar}R\left|\sin\frac{\theta}{2}\right|\ll 1} = \frac{e^2\Phi^2}{8\pi c^2\hbar p\sin^2\frac{\theta}{2}} \qquad \checkmark$$

Remember:

$$\left.\frac{d\sigma}{d\theta}\right|_{AB} = \hbar\,\frac{\sin^2\left(e\Phi/2\hbar c\right)}{2\pi p\sin^2\left(\theta/2\right)}$$

Q-Perturbative analysis 6

Classical Planck's limit:

$$\hbar \rightarrow 0$$

$$\lim_{\hbar \to 0} \frac{d\sigma}{d\theta} = \lim_{\hbar \to 0} \hbar^2 \left(\frac{e\Phi}{2\pi c}\right)^2 \frac{\cos^2\left(2\frac{p}{\hbar}R\left|\sin\left(\theta/2\right)\right| - 3\pi/4\right)}{2R^3p^4\left|\sin^5\left(\theta/2\right)\right|} = 0$$

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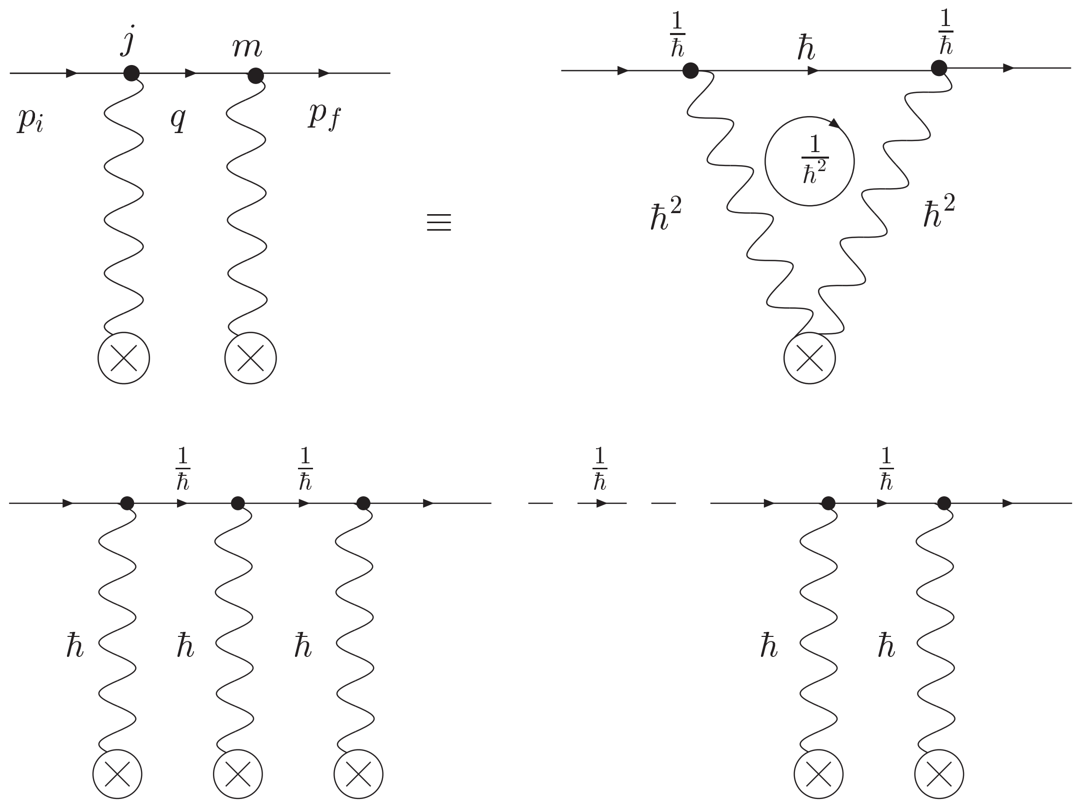
Higher orders?

Possible non-classical asymmetries

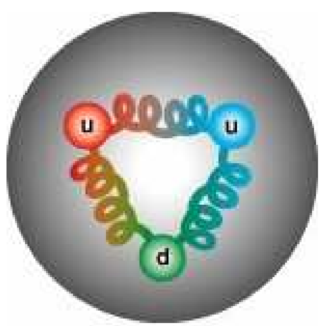
Murguia, Moreno and Torres, **quant-ph/0407123**

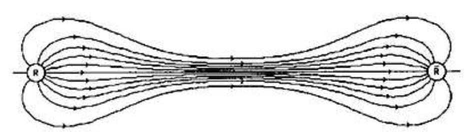


What we understand

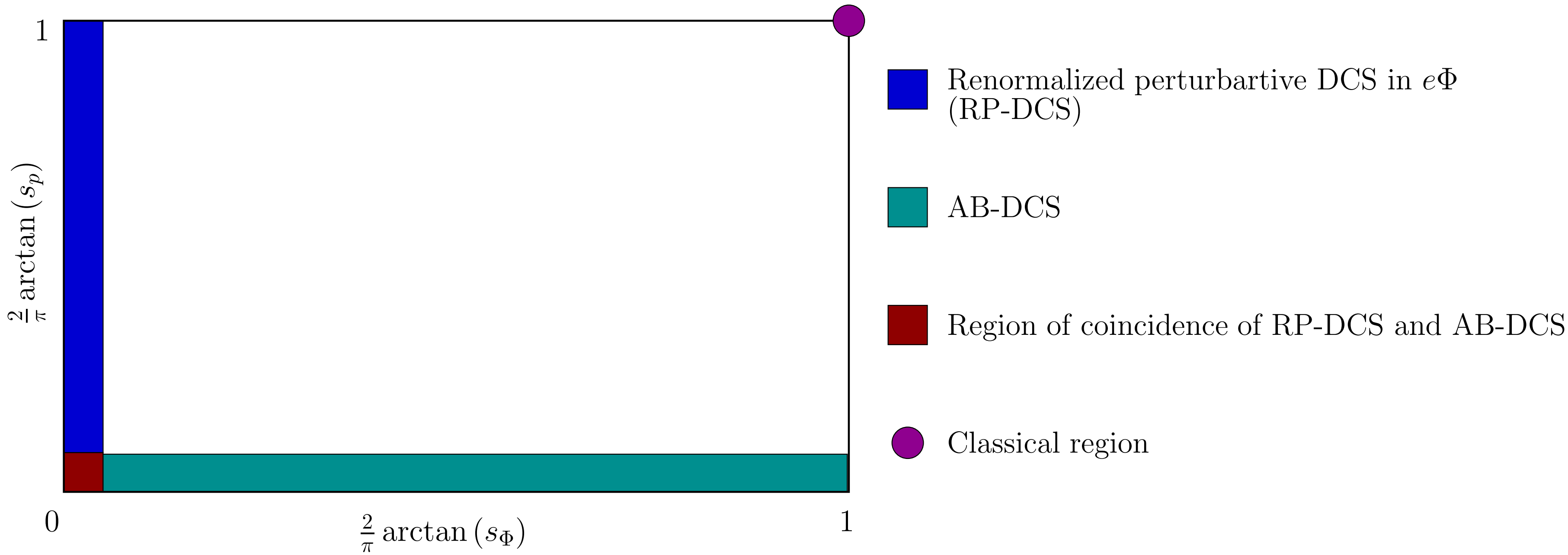
- **New** pure quantum phenomenon
- Perturbative analysis (renormalized to all order)
  $\longrightarrow$ incomplete information
- Asymmetries are not explained by perturbation theory

Example: pQCD





What we understand 2



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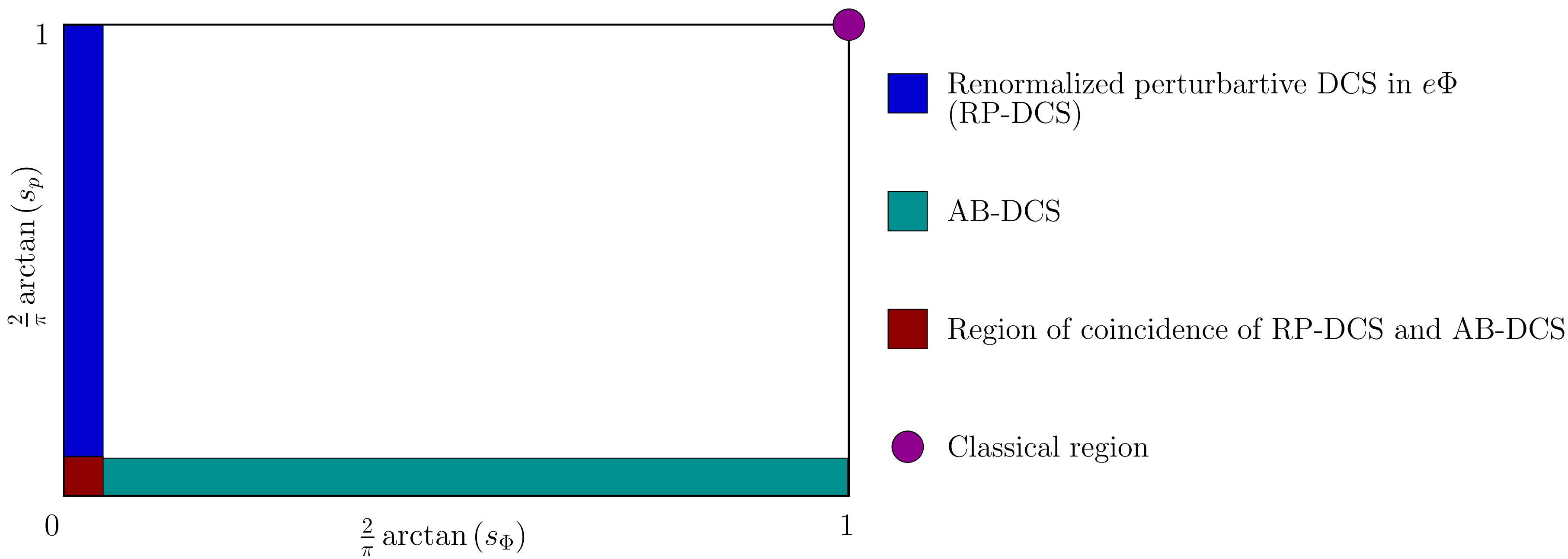
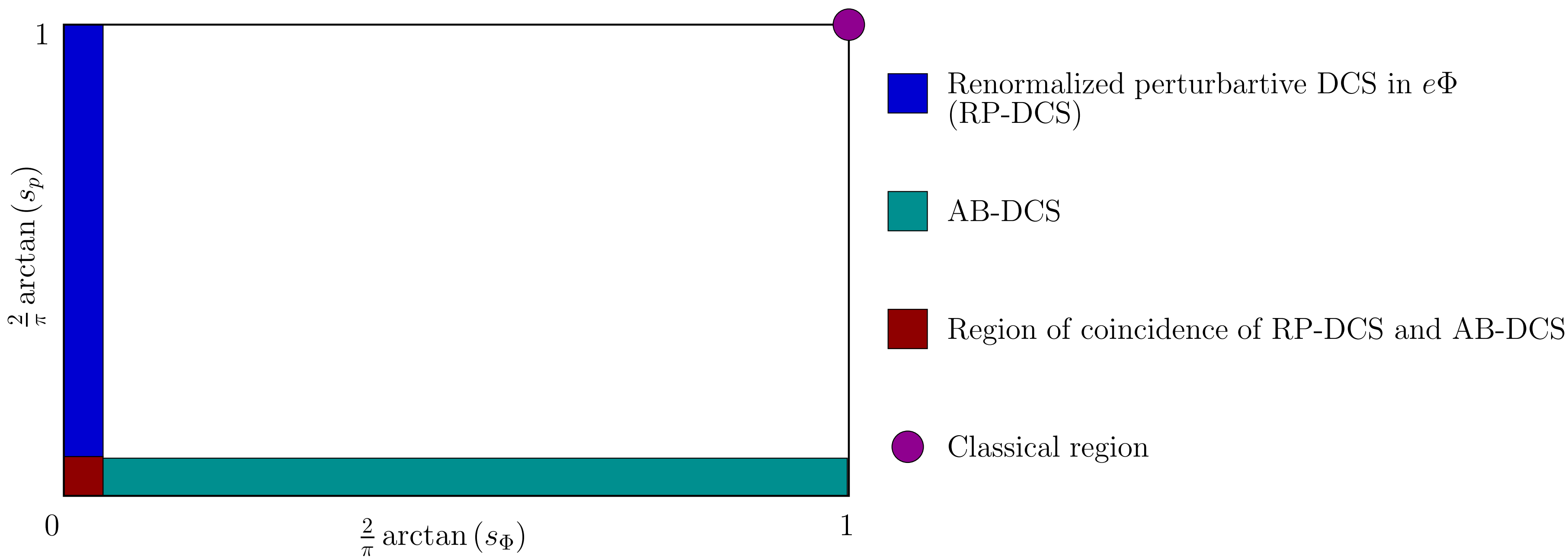


$$\left.\frac{d\sigma}{d\theta}\right|_{AB} = \hbar\,\frac{\sin^2\left(e\Phi/2\hbar c\right)}{2\pi p\sin^2\left(\theta/2\right)}$$

What we understand 2



- $$\frac{d\sigma}{d\theta} = \hbar \left(\frac{e\Phi}{Rc}\right)^2 \frac{\left|J_1\left(2\frac{p}{\hbar}R|\sin(\theta/2)|\right)\right|^2}{8\pi p^3 \sin^4(\theta/2)}$$

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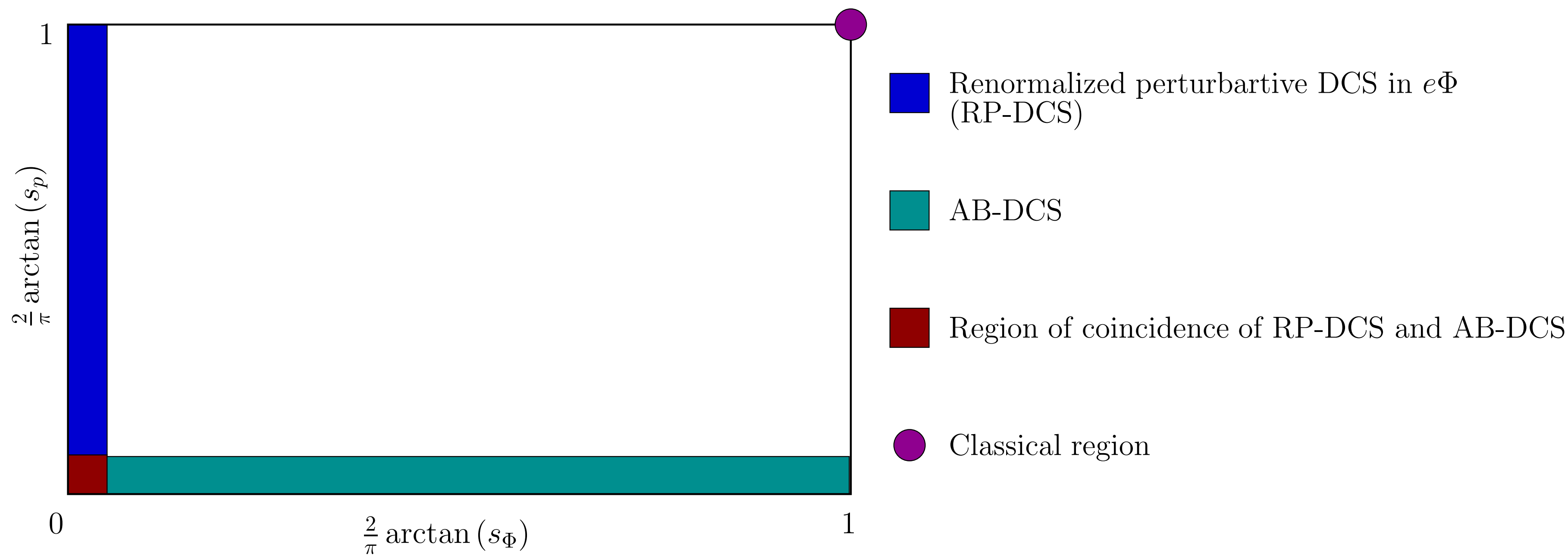

$$\frac{d\sigma}{d\theta}\Big|_{LLP;\theta\ll 1} = \frac{e^2\Phi^2}{2\pi\hbar c^2 p\theta^2}$$

Landau, Lifshitz and Pitaevkii, *Quantum Mechanics (non relativistic theory)*

(Pergamon Press, Oxford) 1977

Q-Exact analysis ($R > 0$)

Ingredients for a non-relativistic analysis:

- Schrödinger's equation $\Phi = (Z + \epsilon)\Phi_0$, $\Phi_0 = hc/e$
- All quantum numbers large as compared with $\hbar$
- Stationary phase approximation

  Berry and Mount, *Rep. Prog. Phys.* **35**, 315 (1972)

  Landau, Lifshitz and Pitaevskii, *Quantum Mechanics (non relativistic theory)* (Pergamon Press, Oxford) 1977

  - We search for ν_0, which corresponds to an extremum of the phase δ_ν ($\nu = L/\hbar$): $2\delta'(\nu_0) + \theta = 0$

Notice: Relativistic and Non-relativistic regimes are related via a Foldy-Wouthuysen transformation

Moreno and Zentella, *J. Phys.* **A22**, L821 (1989)

Q-Exact analysis 2

Scattering Amplitude:

$$f(\theta) = \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} \sum_{m=-\infty}^{m=\infty} e^{im\theta} \left\{ e^{2i\delta_m} - 1 \right\}$$

Classical limit:

$$f(\theta) = \frac{e^{-i\pi/4}}{\sqrt{2\pi k}} e^{i(\nu_0 - 1/2)\theta} e^{2i\delta(\nu_0)} \left[1 + i\,\mathrm{sign}(\delta''(\nu_0))\right] \sqrt{\frac{\pi}{2|\delta''(\nu_0)|}}$$

The differential cross section:

$$\frac{d\sigma}{d\theta} = |f(\theta)|^2 = \frac{1}{2k} \frac{1}{|\delta''(\nu_0)|}$$

Q-Exact analysis 3

Notice: The differential cross section coincides with the classical one

$$L = \hbar\nu \qquad \Rightarrow \qquad b = \nu/k$$

$$\bigoplus$$

$$2\delta'(\nu_0) + \theta = 0$$

implies

$$\left.\frac{d^2\delta}{d\nu^2}\right|_{\nu=\nu_0} = -\frac{1}{2}\frac{d\theta}{d\nu} = -\frac{1}{2k}\frac{d\theta}{db},$$

hence

$$\frac{d\sigma}{d\theta} = \left|\frac{db}{d\theta}\right| = \frac{1}{2k}\,\frac{1}{|\delta''(\nu_0)|}$$

Q-Exact analysis 4

Rigid Solenoid

- Solution to Schrödinger's equation:

$$\psi(\mathbf{r}) = \sum_{m=-\infty}^{m=\infty} e^{im\theta} C_m [J_{m+N}(kr) + D_m N_{m+N}(kr)]$$

- The phase is: $e^{2i\delta_m} = \frac{H^{(2)}_{m+\epsilon}(kR)}{H^{(1)}_{m+\epsilon}(kR)}, \qquad \epsilon \in (0,1)$

Q-Exact analysis 4

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$\surd$ When $R \to 0$, AB limit recovered

$\surd$ Classical limit obtained:

$$\frac{d\sigma}{d\theta} = \frac{R}{2} \left| \operatorname{sen}(\theta/2) \right|$$

Q-Exact analysis 5

Penetrable Solenoid

- Solutions to Schrödinger's equation:

$$\psi_>(\mathbf{r}) = \sum_{m=-\infty}^{m=\infty} e^{im\theta} \left[C_1^> J_{m+N}(k_\perp r) + C_2^> N_{m+N}(k_\perp r)\right]$$

$$\psi_<(\mathbf{r}) = \sum_{m=-\infty}^{m=\infty} e^{im\theta}\, C^< x^{|m|/2} e^{-x/2} M(\alpha/2, \gamma; x)$$

$$x = N\frac{r^2}{R^2}, \quad \alpha = |m| + m + 1 - \frac{2}{N}\left(\frac{kR}{2}\right)^2, \quad \gamma = |m| + 1$$

Q-Exact analysis 6

- The phase is:

$$e^{2i\delta_m} = -\frac{H_m^{(2)}(kR)\mathcal{M}(m-N,N,kR) - H_m^{(2)'}(kR)}{H_m^{(1)}(kR)\mathcal{M}(m-N,N,kR) - H_m^{(1)'}(kR)}$$

$$\mathcal{M}(m,N,kR) \equiv \frac{1}{kR}\left[|m| - N + N\,\frac{\alpha}{\gamma}\,\frac{M(\alpha/2+1,\gamma+1;N)}{M(\alpha/2,\gamma;N)}\right]$$

Q-Exact analysis 6

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Q-Exact analysis 7

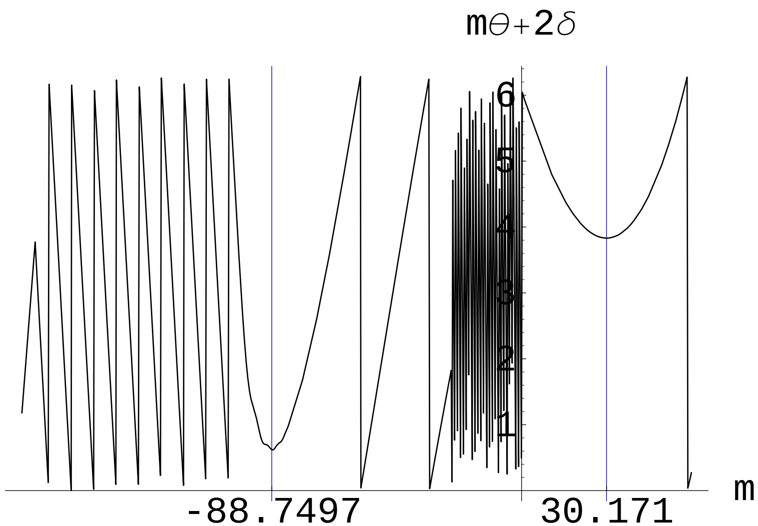
$m\,\theta + 2\,\delta(m)$ vs m

$$\kappa \equiv kR = 100$$

$$(b = L/p = m/k \Rightarrow \kappa \cdot \rho_b = k \cdot b = m)$$

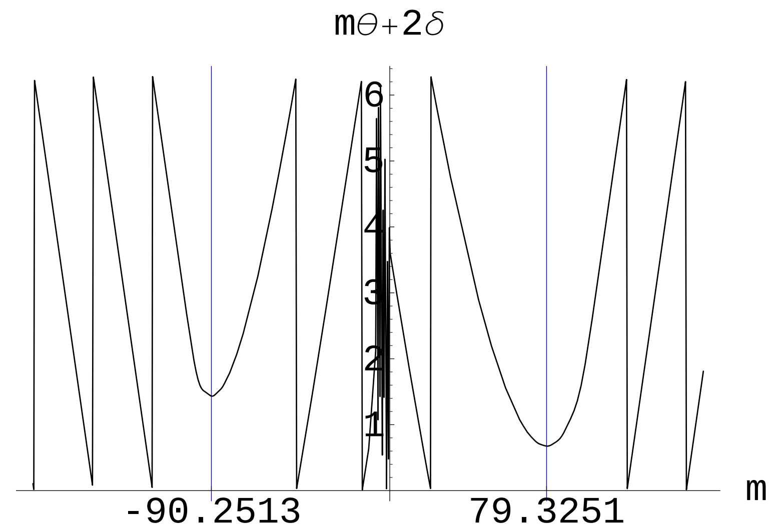
$\theta = \pi/4 \qquad N = 25$

$\rho_L = 2.0 \qquad \kappa \cdot \rho_b = 30.17, -88.75$



$\theta = \pi/15 \qquad N = 10$

$\rho_L = 5.0 \qquad \kappa \cdot \rho_b = 79.32, -90.25$



Q-Exact analysis 8

$m\,\theta + 2\,\delta(m)$ vs m

$$\kappa \equiv kR = 100$$

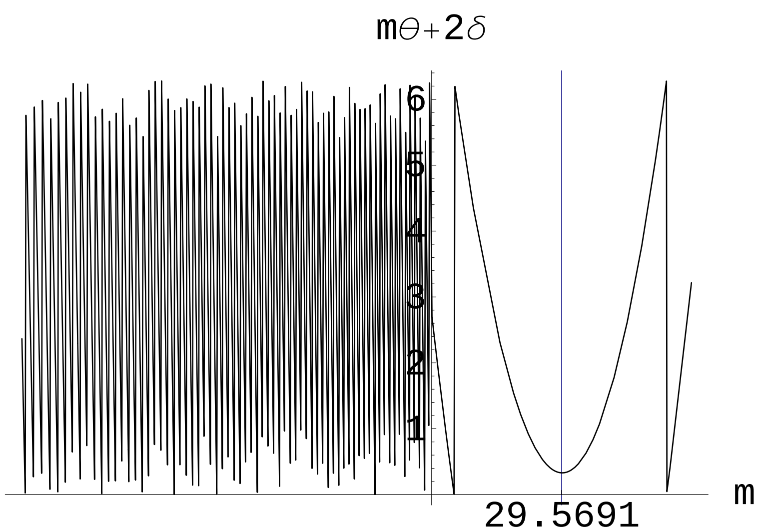
$$(b = L/p = m/k \Rightarrow \kappa \cdot \rho_b = k \cdot b = m)$$

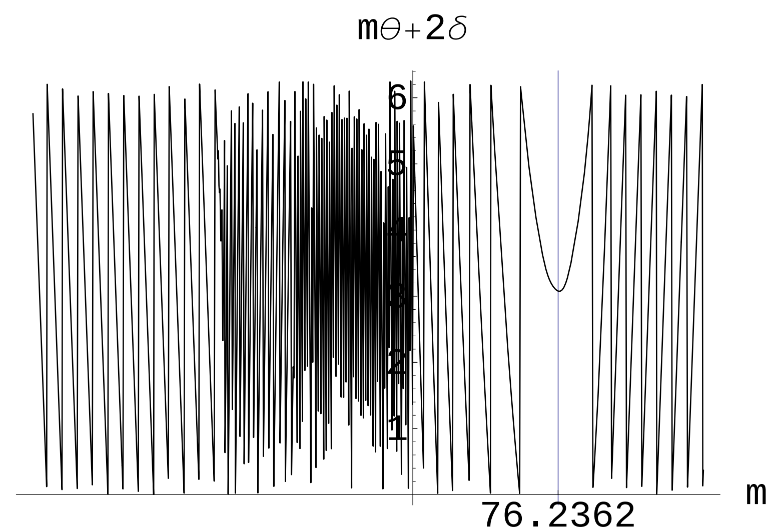
$\theta = 3\pi/4 \qquad N = 500$

$\rho_L = 0.1 \qquad \kappa \cdot \rho_b = 29.57$

$\theta = \pi/4 \qquad N = 62.5$

$\rho_L = 0.8 \qquad \kappa \cdot \rho_b = 76.24$





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- $d\sigma$ (relativistic) perturbative, symmetric to lowest order in e^2
- Non-classical asymmetries
- Classical limit not recovered
- **New** quantum effect

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- $d\sigma$ asymmetric and finite. $\sigma = 2R$

Quantum regime:

- $d\sigma$ (relativistic) perturbative, symmetric to lowest order in e^2
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- Classical limit not recovered
- **New** quantum effect

- $d\sigma$ (non-relativistic) exact, asymmetric
- Classical limit recovered