Perturbative Quantum Analysis and Classical Limit of the Electron Scattering by a Solenoidal Magnetic Field

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Abstract.
A well known example in quantum electrodynamics (QED) shows that Coulomb scattering of unpolarized electrons, calculated to lowest order in perturbation theory, yields a results that exactly coincides (in the non-relativistic limit) with the Rutherford formula. We examine an analogous example, the classical and perturbative quantum scattering of an electron by a magnetic field confined in an infinite solenoid of finite radius. The results obtained for the classical and the quantum differential cross sections display marked differences. While this may not be a complete surprise, one should expect to recover the classical expression by applying the classical limit to the quantum result. This turn not to be the case. Surprisingly enough, it is shown that the classical result can not be recuperated even if higher order corrections are included. To recover the classic correspondence of the quantum scattering problem a suitable non-perturbative methodology should be applied.

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INTRODUCTION

As it is widely known, the scattering of unpolarized electrons by the Coulomb potential exactly coincides with the classical Rutherford formula, if one consider the lowest order in perturbation theory and the non-relativistic regime. In this paper, we examine an analogous example: the scattering of an electron of momentum p by a magnetic field in a long solenoid of fixed flux Φ and finite radius R, looking both at the classical and quantum regimes.

In the zero radius limit, the differential cross section (DCS) is given by the famous Aharonov-Bohm (AB) result [1]:

$$\left.\frac{d\sigma}{d\theta}\right|_{AB} = \hbar\,\frac{\sin^2\left(e\Phi/2\hbar c\right)}{2\pi p \sin^2\left(\theta/2\right)}. \tag{1}$$

This is a purely quantum effect, because in the $\hbar \to 0$ limit the expression cancels.

For a finite value R of the solenoid radius, the classical cross section will have a definite non-vanishing value, as far as the electron can penetrate inside the solenoid. We shall calculate the expression for this classical cross section. One may wonder if there is a connection between the quantum and classical regimes for finite solenoid radius. We find that the differential cross section obtained from the first order QED calculation does not reduce to the classical value in the $\hbar \to 0$ limit. Surprisingly enough, it is shown that the classical result can not be recuperated even if higher order corrections are included. To recover the classic correspondence of the quantum scattering problem a suitable non-perturbative methodology should be applied.

CLASSICAL CROSS SECTION

Let us first consider the classical differential cross section of charged particles by the magnetic field of a long solenoid of finite radius R and fixed magnetic flux Φ. Utilizing the classical equation of motion the scattering angle as a function of the impact parameter b is obtained as

$$\theta(b) = 2\arctan\left(\frac{\sqrt{R^2-b^2}}{b+r_L}\right), \tag{2}$$

where $r_L = pc/eB$ is the Larmor radius. The impact parameter $b(\theta)$ is a multiple-valued function of θ; hence, the differential cross section requires to add the two branches of the function, the result is worked out as

$$\begin{aligned}\frac{1}{R}\frac{d\sigma_{\rho_L}(\theta)}{d\theta} &= \left|\frac{\sin\theta}{2}\left(\rho_L + \frac{1+\rho_L^2\cos\theta}{2\cos(\theta/2)\sqrt{1-\rho_L^2\sin^2(\theta/2)}}\right)\right| + \\ &\quad \left|\frac{\sin\theta}{2}\left(\rho_L - \frac{1+\rho_L^2\cos\theta}{2\cos(\theta/2)\sqrt{1-\rho_L^2\sin^2(\theta/2)}}\right)\right|\Theta(|\rho_L|-1), \end{aligned}\tag{3}$$

where we defined a dimensionless parameter $\rho_L = r_L/R = pR/2\beta$, with $\beta = e\Phi/2\pi c$; and $\Theta(x)$ is the Heaviside step function. Notice that in the low energy regime ($\rho_L < 1$) the scattering angle covers all the range $\theta \in [-\pi,\pi)$. Instead for $\rho_L > 1$ there is a maximum allowed scattering angle $\theta \in [0,\theta_{\max}]$; where $\sin(\theta_{\max}) = 1/\rho_L$.

As expected, the Lorentz's force produces in general a classical DCS that is not symmetric with respect to the forward direction ($\theta = 0$). Furthermore, it is worthwhile to observe the highly nonlinear dependence of the DCS on the coupling $\beta = e\Phi/2\pi c$.

The impenetrable limit $\rho_L \to 0$ is obtained with $pR \to 0$ and fixed Φ; or considering the limit $\Phi \to \infty$ with fixed pR; in both case the DCS reduces to

$$\left.\frac{d\sigma}{d\theta}\right|_{\rho_L\to 0} = \frac{R}{2}\left|\sin(\theta/2)\right|, \tag{4}$$

a result that, as expected, is symmetric with respect to the forward direction and independent of the coupling to the magnetic field.

Another interesting limit is obtained for high energy incident particles with fixed magnetic flux: $pR \to \infty$ ($\rho_L \gg 1$). The scattered electrons are confined inside a narrow cone aligned along the forward direction, defined by the maximum angle $\theta_{\rm max} \approx 1/\rho_L$. It is possible to show from equation (3) that the cross section reduces to

$$\left.\frac{d\sigma}{d\theta}\right|_{\rho_L \gg 1} \approx R\,\theta \frac{1+\rho_L^2}{\sqrt{4-\rho_L^2\theta^2}}, \qquad |\theta| \leq \theta_{\rm max}. \tag{5}$$

We notice again the nonlinear dependence of the DCS on the coupling $e\Phi$, a result that anticipates the incompatibility of the classical result with the one that will be obtained in a quantum perturbative calculation to any given finite order.

PERTURBATIVE QUANTUM ANALYSIS

We now turn our attention to the calculation of the DCS in the quantum regime. The electron interacts with the gauge potential, that for the finite radius solenoid can be represented as

$$A_i = -\frac{\Phi}{2\pi}\varepsilon_{ij3}x_j \begin{cases} \frac{1}{R^2} & \text{for } r < R \\ \frac{1}{x_1^2+x_2^2} & \text{for } r > R. \end{cases} \tag{6}$$

The interaction of the electron with the external magnetic field is taken into account by introducing a dimensionless coupling factor $e\Phi/\hbar c$ for each interaction of the electron with the external field, and a factor related to the Fourier transformation of the gauge potential A_i:

$$-2i\frac{\hbar^2}{R}J_1(q_\perp R/\hbar)\varepsilon_{ij3}\frac{q_j}{q_\perp^3}, \tag{7}$$

where $q_\perp$ refers to the momentum perpendicular to the direction of the magnetic field, and J_1 is the Bessel functions of first kind.

The DCS was calculated in reference [2] to the lowest perturbative order in $\beta = e\Phi/2\pi c$, using free particle incident and final asymptotic states, yielding

$$\frac{d\sigma}{d\theta} = \hbar\left(\frac{e\Phi}{Rc}\right)^2 \frac{\left|J_1(2\frac{p}{\hbar}R|\sin(\theta/2)|)\right|^2}{8\pi p^3\sin^4(\theta/2)}. \tag{8}$$

The previous result has the same form whether or not the final polarization of the beam is actually measured. As can be observed, the cross section is symmetric in the scattering angle θ with respect to the forward direction.

The marked different behavior between the classical and quantum DCS becomes evident; first from the symmetric behavior of the quantum result, equation (8), as compared to the asymmetric structure of the classical one, equation (3). Furthermore, notice that the total quantum cross section is infinite, in contrast to the finite value of $2R$ obtained for the classical case. More important is the fact that the quantum DCS in equation (8) is directly proportional to the coupling $e\Phi$, while the classical DCS diverges as $e\Phi \to 0$.

In order to consider the classical limit of the DCS in equation (8), we recall that according to Berry and Mount [3] and Gutzwiller [4], the implementation of the classical limit requires to look at the situation in which the action quantities that appear in the corresponding classical problem are considered as very large as compared to $\hbar$ [5]. Here we identify two action variables, selected as: pR and $e\Phi/c$. It is then convenient to define the dimensionless parameters $s_p = pR/\hbar$ and $s_\Phi = e\Phi/\hbar c$. In term of these dimensionless parameters the DCS in equation (8) can be recast as

$$\frac{d\sigma}{d\theta} = \frac{R}{8\pi}\frac{s_\Phi^2}{s_p^3}\left|\frac{J_1(2s_p|\sin(\theta/2)|)}{\sin^2(\theta/2)}\right|^2. \tag{9}$$

The classical limit is enforced by considering both $s_p \gg 1$ and $s_\Phi \gg 1$. We observe that the classical limit of the DCS in equation (9) vanishes because it behaves as $s_\Phi^2/s_p^4 \propto \hbar^2 \to 0$. This result establishes that the classical DCS can not be recovered in the "classical limit" of the quantum DCS calculated to first order in $\beta = e\Phi/2\pi c$.

Higher order processes can be calculated using the Feynman rules for the electron-solenoid scattering [6]. Counting the $\hbar$ power contributions to higher order diagrams (as the one depicted in figure 1) and assuming free particle asymptotic states, it can be shown that higher orders in β do not modify the leading $\hbar$ power contribution to the scattering matrix; in fact higher order corrections in β contribute with terms proportional to positive higher powers in $\hbar$.

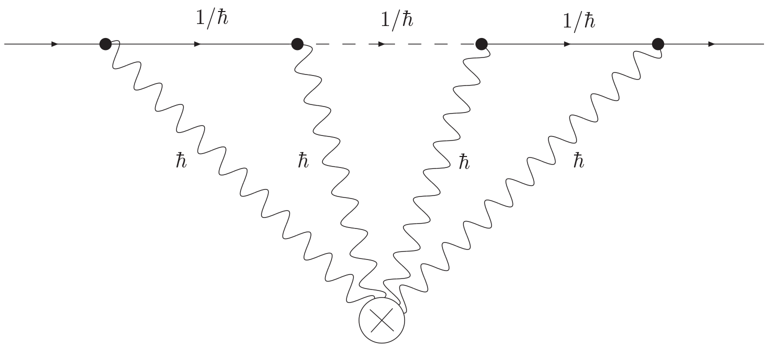

FIGURE 1. Feynman diagram and $\hbar$ power counting for an arbitrary order in $\beta = e\Phi/2\pi c$ of the scattering matrix for a solenoidal magnetic field. The wiggled lines represent the interaction with the external magnetic field while the straight lines represent the free-fermion propagators.

We recall that usual radiative corrections (higher powers in α) will in general contribute with positive $\hbar$ powers to the matrix elements, hence they are not expected to be relevant in the classical limit. Consequently, for arbitrary finite order the perturbative expansion in both β and α produces a contribution proportional to powers of $\hbar$, that cancels in the classical limit of this process. Consequently the classical expression for the DCS can not be recovered.

The various regions for the scattering electron-solenoid process are schematically displayed in the diagram of figure 2. For illustrative purposes, the arc-tangent of s_p and s_Φ are normalized to unity. There are depicted the regions in which equations (1) and (8) are valid, including the renormalized perturbative terms in $\beta = e\Phi/2\pi c$. Notice that the Aharonov-Bohm DCS is valid for small s_p; whereas the perturbative results in β are valid in the small s_Φ region. Both results coincide in the $s_p \to 0$ and $s_\Phi \to 0$ region [2]. It is expected that the exact quantum calculation (valid for all values of s_p and s_Φ) has the correct classical limit in the $s_p \to \infty$ and $s_\Phi \to \infty$ region, which is depicted with a dot in the upper right corner of the diagram.

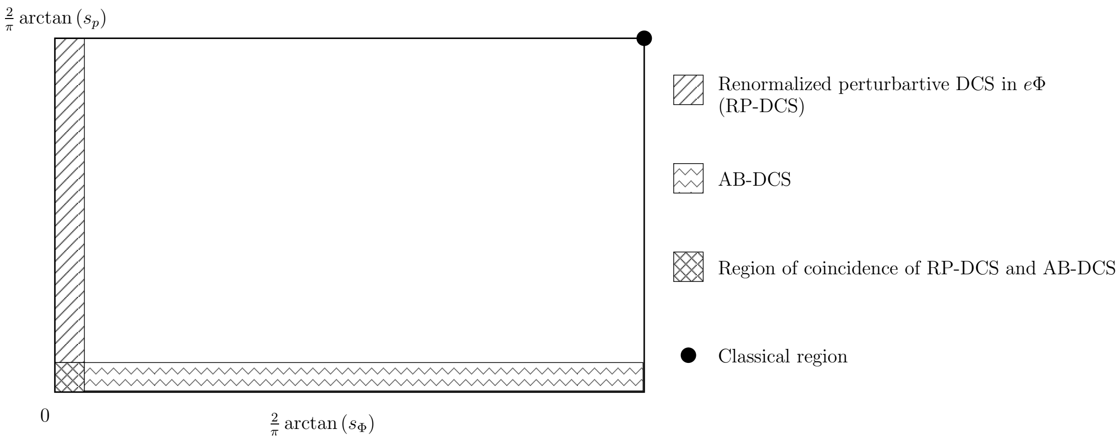

FIGURE 2. Diagram s_p vs s_Φ for the quantum cross section of the scattering by a solenoidal magnetic field. The results for small s_p and s_Φ are shown by the dashed regions. The classical region is represented by the dot in the upper right corner.

CONCLUSIONS

In this paper we have studied the classical and quantum scattering of an electron by a magnetic field confined in an infinite solenoid of finite radius. In the classical scenario the DCS shows a nonlinear dependence on the coupling parameter $\beta = e\Phi/2\pi c$ and a general asymmetric behavior with respect to the forward direction. The DCS obtained in the perturbative quantum regime displays marked differences as compared with the classical one. The classical limit of a corresponding quantum observable is characterized as the limit in which all the relevant action quantities are considered very large as compared with $\hbar$. We found that the classical DCS is not recovered from the quantum DCS, even if higher order corrections are included. We conclude that in general perturbative calculations easily could drive to unappropriated results in the classical limit, because in the perturbative regime typically at least one parameter remains small in comparison with $\hbar$.

To recover the classical correspondence of the quantum scattering problem a suitable non-perturbative methodology should be applied. It has been shown in [7] that an exact expression for the quantum non-relativistic DCS can be obtained. Then a combination of the large action_variable$/\hbar$ limit, with an stationary phase approximation for the evaluation of the partial wave summation can be successfully implemented in order to correctly derive the classical limit .

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