On the Condensed Matter Analog of Baryon Chiral Perturbation Theory

Christoph P. Hofmann

In collaboration with C. Brügger, F. Kämpfer, M. Moser, M. Pepe and U.-J. Wiese

Facultad de Ciencias, UCOL

October 10, 2008

Outline

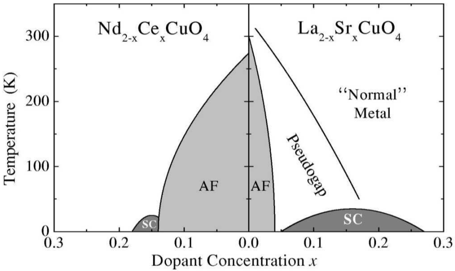
Outline

Motivation: High-T_c superconductivity in cuprates

1986: Bednorz and Müller discover high-T_c superconductivity by doping copper oxide compounds (cuprates):

$$La_2CuO_4 \longrightarrow La_{2-x}Ba_xCuO_4 \qquad (T_c = 35\ \mathrm{K})$$

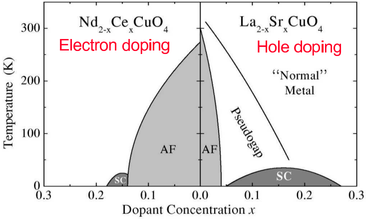
Motivation: High-T_c superconductivity in cuprates



- SC results from doping antiferromagnetic insulators
- Doping possible with both electrons and holes
- Electron-Hole asymmetry

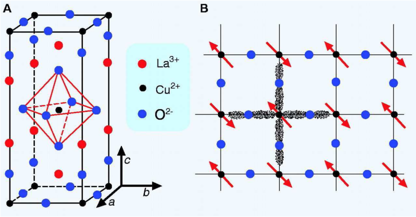
Motivation: High-T_c superconductivity in cuprates

Phase diagram of cuprates:



Damascelli, Hussain, and Shen, Rev. Mod. Phys. 75 (2003) 473

Crystal structure:

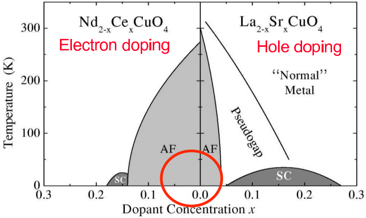


Orenstein and Millis, Science 288 (2000) 468

- SC results from doping antiferromagnetic insulators
- Doping possible with both electrons and holes
- Common structure: CuO_2 layers separated by spacer layers

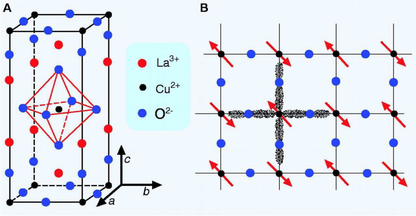
Motivation: High-T_c superconductivity in cuprates

Phase diagram of cuprates:



Damascelli, Hussain, and Shen, Rev. Mod. Phys. 75 (2003) 473

Crystal structure:



Orenstein and Millis, Science 288 (2000) 468

- SC results from doping antiferromagnetic insulators
- Doping possible with both electrons and holes
- Common structure: CuO_2 layers separated by spacer layers
- Concentrate on antiferromagnetic region: low doping, low T

Microscopic description: The Hubbard model

The Hubbard Hamiltonian defined on a square lattice:

$$H = -t \sum_{x,i} (c_{x\uparrow}^\dagger c_{x+\hat{i}\uparrow} + c_{x+\hat{i}\uparrow}^\dagger c_{x\uparrow} + c_{x\downarrow}^\dagger c_{x+\hat{i}\downarrow} + c_{x+\hat{i}\downarrow}^\dagger c_{x\downarrow}) + U \sum_x c_{x\uparrow}^\dagger c_{x\uparrow} c_{x\downarrow}^\dagger c_{x\downarrow} - \mu \sum_x (c_{x\uparrow}^\dagger c_{x\uparrow} + c_{x\downarrow}^\dagger c_{x\downarrow} - 1)$$

- Parameters:

t : Hopping parameter (nearest neighbors)

U : On-site Coulomb repulsion

μ : Chemical potential for fermion number

Microscopic description: The Hubbard model

The Hubbard Hamiltonian defined on a square lattice:

$$H = -\,t\sum_{x,i}(c^\dagger_{x\uparrow}c_{x+\hat{i}\uparrow} + c^\dagger_{x+\hat{i}\uparrow}c_{x\uparrow} + c^\dagger_{x\downarrow}c_{x+\hat{i}\downarrow} + c^\dagger_{x+\hat{i}\downarrow}c_{x\downarrow})$$
$$+\,U\sum_{x} c^\dagger_{x\uparrow}c_{x\uparrow}c^\dagger_{x\downarrow}c_{x\downarrow} - \mu\sum_{x}(c^\dagger_{x\uparrow}c_{x\uparrow} + c^\dagger_{x\downarrow}c_{x\downarrow} - 1)$$

- Parameters:

$$\begin{aligned} t &: \text{ Hopping parameter (nearest neighbors)} \\ U &: \text{ On-site Coulomb repulsion} \\ \mu &: \text{ Chemical potential for fermion number} \end{aligned}$$

- Minimal model for cuprates: contains the relevant physics
- Away from half-filling: Hamiltonian virtually unsolvable from first principles (Neither analytically nor numerically)

Microscopic description: The Hubbard model

The Hubbard Hamiltonian defined on a square lattice:

$$H = -t \sum_{x,i} (c_{x\uparrow}^\dagger c_{x+\hat{i}\uparrow} + c_{x+\hat{i}\uparrow}^\dagger c_{x\uparrow} + c_{x\downarrow}^\dagger c_{x+\hat{i}\downarrow} + c_{x+\hat{i}\downarrow}^\dagger c_{x\downarrow})$$
$$+ U \sum_x c_{x\uparrow}^\dagger c_{x\uparrow} c_{x\downarrow}^\dagger c_{x\downarrow} - \mu \sum_x (c_{x\uparrow}^\dagger c_{x\uparrow} + c_{x\downarrow}^\dagger c_{x\downarrow} - 1)$$

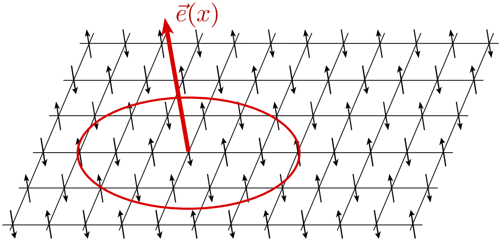
- Symmetries:

$$\begin{aligned} SU(2)_s &: \text{ Global spin rotation} \\ U(1)_Q &: \text{ Fermion number conservation} \\ D_i &: \text{ Displacement by one lattice spacing} \\ O &: \text{ 90 degrees rotation} \\ R &: \text{ Reflection on a lattice axis} \\ T &: \text{ Time reversal} \end{aligned}$$

Antiferromagnetism: Near half-filling (1 fermion per site)

Near half-filling:

- Antiferromagnetic alignment of spins is preferred
- Spontaneous symmetry breaking: $SU(2)_s \longrightarrow U(1)_s$
- Goldstone's theorem: 2 massless excitations $\Longrightarrow$ 2 magnons

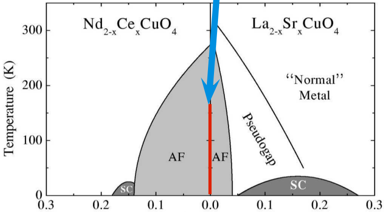


Systematic effective field theory description

	Antiferromagnets	QCD
Spont. symm. breaking	$SU(2)_s \longrightarrow U(1)_s$	$SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_{L=R}$
GB physics	Magnon perturbation theory	Chiral perturbation theory
GB + matter physics	Effective theory presented here	Baryon chiral perturbation theory

Systematic effective field theory description

	Antiferromagnets	QCD
Spont. symm. breaking	$SU(2)_s \longrightarrow U(1)_s$	$SU(2)_L \otimes SU(2)_R \rightarrow SU(2)_{L=R}$
GB physics	Magnon perturbation theory	Chiral perturbation theory
GB + matter physics	Effective theory presented here	Baryon chiral perturbation theory



Pure magnon sector: Magnon perturbation theory

Spontaneous global $SU(2)_s \longrightarrow U(1)_s$ spin symmetry breaking:

- 2 Goldstone bosons (magnons) described by

$$\vec{e}(x) = \big(e_1(x), e_2(x), e_3(x)\big) \in S^2 = SU(2)_s/U(1)_s$$

with $x = (x_1, x_2, t)$

Pure magnon sector: Magnon perturbation theory

Spontaneous global $SU(2)_s \longrightarrow U(1)_s$ spin symmetry breaking:

- 2 Goldstone bosons (magnons) described by

$$\vec{e}(x) = \big(e_1(x), e_2(x), e_3(x)\big) \in S^2 = SU(2)_s/U(1)_s$$

  with $x = (x_1, x_2, t)$

- Low-energy magnon physics described by nonlinear σ-model

$$\mathcal{L} = \frac{\rho_s}{2}(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2}\partial_t \vec{e} \cdot \partial_t \vec{e}) + \cdots$$

ρ_s: spin stiffness c: spin wave velocity

Chakravarty, Halperin, and Nelson, PRB 39 (1989) 2344
Hasenfratz and Niedermayer, Phys. Lett. B268 (1991) 231

Outline

State of affairs: Fermionic sector of effective theory

Earlier attempts by: Shraiman and Siggia, Wen, Shankar, ...

General agreement:

- Magnons are coupled to fermions through composite vector fields

State of affairs: Fermionic sector of effective theory

Earlier attempts by: Shraiman and Siggia, Wen, Shankar, ...

General agreement:

- Magnons are coupled to fermions through composite vector fields

No agreement on low-energy effective Lagrangian for fermions:

- Conflicting realizations of fermion fields
- Non-unique structure of terms in Lagrangians

$\Longrightarrow$ Model Lagrangians have not been constructed systematically

State of affairs: Fermionic sector of effective theory

Earlier attempts by: Shraiman and Siggia, Wen, Shankar, ...

General agreement:

- Magnons are coupled to fermions through composite vector fields

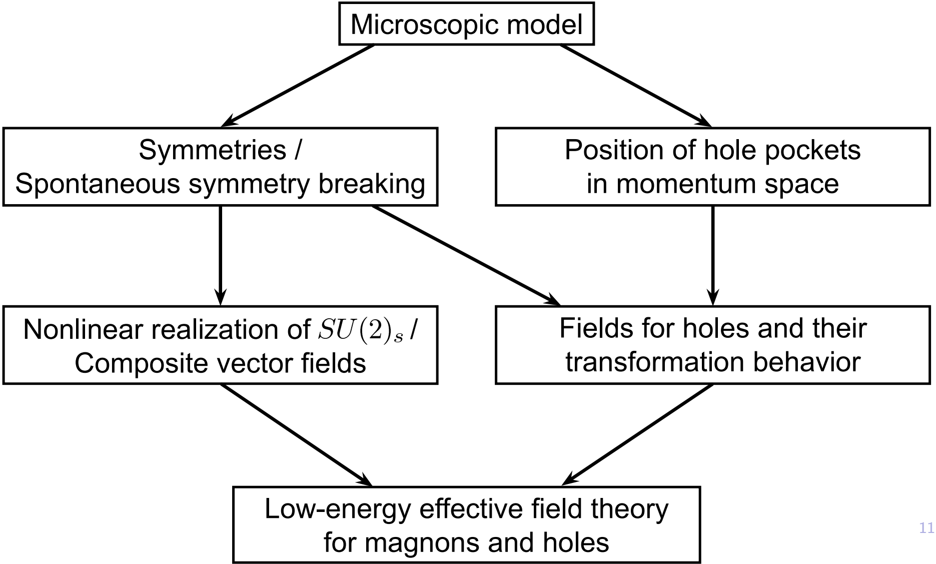
No agreement on low-energy effective Lagrangian for fermions:

- Conflicting realizations of fermion fields
- Non-unique structure of terms in Lagrangians

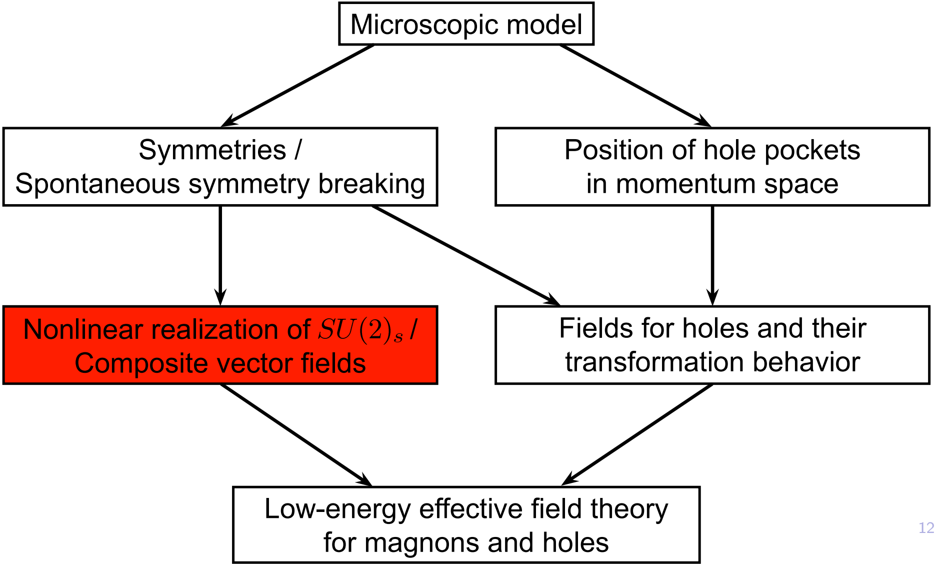
$\Longrightarrow$ Model Lagrangians have not been constructed systematically

$\Longrightarrow$ Construction of a systematic low-energy effective field theory for magnons and holes analogous to baryon chiral perturbation theory

Symmetry-based construction of effective theory



Symmetry-based construction of effective theory



Nonlinear realization of $SU(2)_s$ symmetry

- $\mathbb{C}P(1)$ representation of magnon field

$$P(x) = \frac{1}{2}\left(\mathbb{1} + \vec{e}(x) \cdot \vec{\sigma}\right) \in \mathbb{C}P(1)$$

Nonlinear realization of $SU(2)_s$ symmetry

- $\mathbb{C}P(1)$ representation of magnon field

$$P(x) = \frac{1}{2}\left(\mathbb{1} + \vec{e}(x) \cdot \vec{\sigma}\right) \in \mathbb{C}P(1)$$

- Diagonalize the magnon field

$$u(x)P(x)u(x)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u(x) \in SU(2)_s, \qquad u_{11}(x) \geq 0$$

Nonlinear realization of $SU(2)_s$ symmetry

- $\mathbb{C}P(1)$ representation of magnon field

$$P(x) = \frac{1}{2}\left(\mathbb{1} + \vec{e}(x) \cdot \vec{\sigma}\right) \in \mathbb{C}P(1)$$

- Diagonalize the magnon field

$$u(x)P(x)u(x)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u(x) \in SU(2)_s, \qquad u_{11}(x) \geq 0$$

- Under global $SU(2)_s$ spin transformations

$$P(x)' = gP(x)g^\dagger, \qquad g \in SU(2)_s$$

Nonlinear realization of $SU(2)_s$ symmetry

- $\mathbb{C}P(1)$ representation of magnon field

$$P(x) = \frac{1}{2}(\mathbb{1} + \vec{e}(x) \cdot \vec{\sigma}) \in \mathbb{C}P(1)$$

- Diagonalize the magnon field

$$u(x)P(x)u(x)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u(x) \in SU(2)_s, \qquad u_{11}(x) \geq 0$$

- Under global $SU(2)_s$ spin transformations

$$P(x)' = gP(x)g^\dagger, \qquad g \in SU(2)_s$$

- The diagonalizing field $u(x)$ transforms as

$$u(x)' = h(x)u(x)g^\dagger, \qquad h(x) \in U(1)_s, \qquad u_{11}(x)' \geq 0$$

Nonlinear realization of $SU(2)_s$ symmetry

- $\mathbb{C}P(1)$ representation of magnon field

$$P(x) = \frac{1}{2}\left(\mathbb{1} + \vec{e}(x) \cdot \vec{\sigma}\right) \in \mathbb{C}P(1)$$

- Diagonalize the magnon field

$$u(x)P(x)u(x)^\dagger = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \qquad u(x) \in SU(2)_s, \qquad u_{11}(x) \geq 0$$

- Under global $SU(2)_s$ spin transformations

$$P(x)' = gP(x)g^\dagger, \qquad g \in SU(2)_s$$

- The diagonalizing field $u(x)$ transforms as

$$u(x)' = h(x)u(x)g^\dagger, \qquad h(x) \in U(1)_s, \qquad u_{11}(x)' \geq 0$$

Global $SU(2)_s$ rotation manifests itself as local $U(1)_s$ transformation!

Composite vector fields

- We introduce an anti-Hermitean field

$$v_\mu(x) = u(x)\partial_\mu u(x)^\dagger = \begin{pmatrix} v_\mu^3(x) & v_\mu^+(x) \\ v_\mu^-(x) & -v_\mu^3(x) \end{pmatrix}$$

  with $\mu \in \{1, 2, t\}$

- Components used to couple magnons to holes

Composite vector fields

- We introduce an anti-Hermitean field

$$v_\mu(x) = u(x)\partial_\mu u(x)^\dagger = \left(\begin{array}{cc} v_\mu^3(x) & v_\mu^+(x) \\ v_\mu^-(x) & -v_\mu^3(x) \end{array} \right)$$

with $\mu \in \{1, 2, t\}$

- Components used to couple magnons to holes

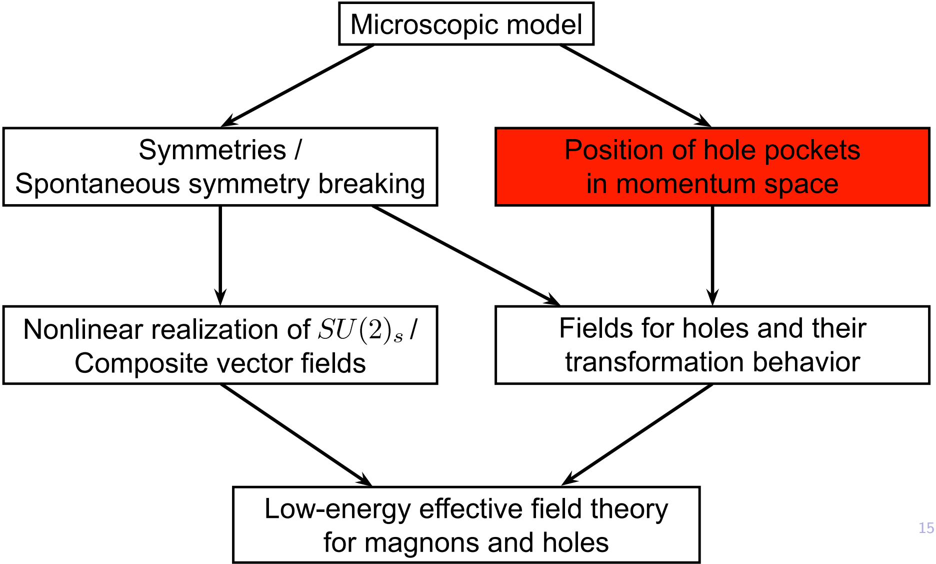
- Under global $SU(2)_s$ the components transform as

$$v_\mu^3(x)' = v_\mu^3(x) - \partial_\mu \alpha(x), \qquad v_\mu^\pm(x)' = v_\mu^\pm(x) \exp\left(\pm 2i\alpha(x)\right)$$

$v_\mu^3(x)$: Abelian gauge field

$v_\mu^\pm(x)$: Vector field ("charged" under $U(1)_s$)

Symmetry-based construction of effective theory



Hole pockets $\Longleftrightarrow$ Effective fields for holes

Where in momentum space do doped holes reside?

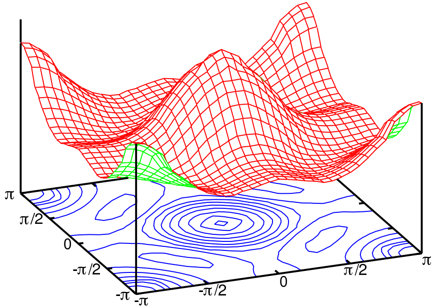
$\Longrightarrow$ Angle resolved photoemission spectroscopy (ARPES)
$\Longrightarrow$ Numerical simulations of single hole in AF

Hole pockets $\Longleftrightarrow$ Effective fields for holes

Where in momentum space do doped holes reside?

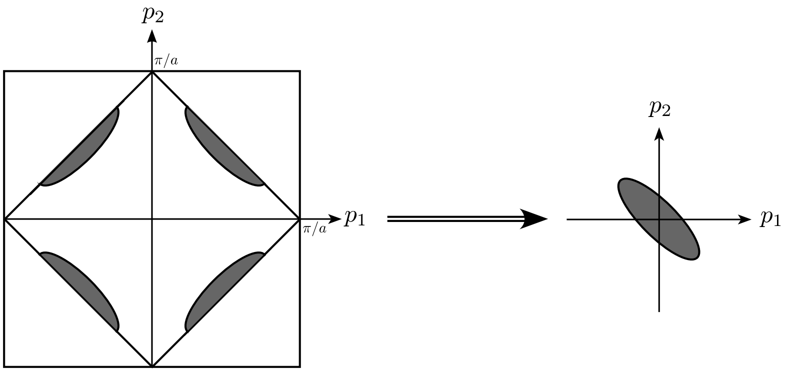
$\Longrightarrow$ Angle resolved photoemission spectroscopy (ARPES)
$\Longrightarrow$ Numerical simulations of single hole in AF

Single hole (away from half-filling) dispersion relation in the first Brillouin zone:



Minima at lattice momenta $\vec{k} = \pm(\frac{\pi}{2a}, \pm\frac{\pi}{2a})$

Hole pockets $\Longleftrightarrow$ Effective fields for holes



Microscopic theory/ ARPES

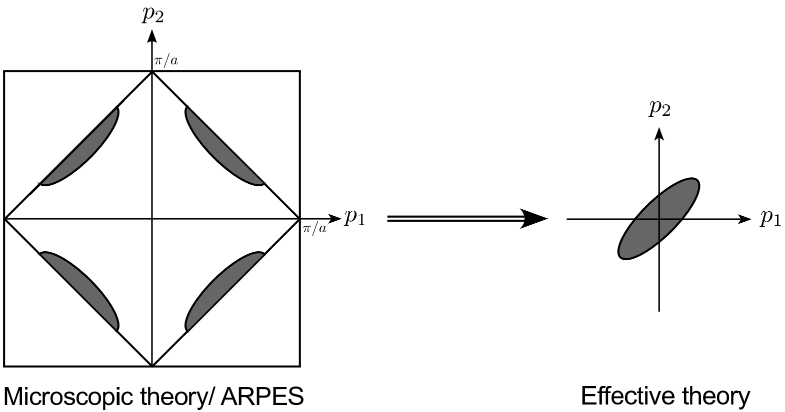
Effective theory

Symmetry considerations: Two half-pockets combine to a full pocket

In the effective theory:

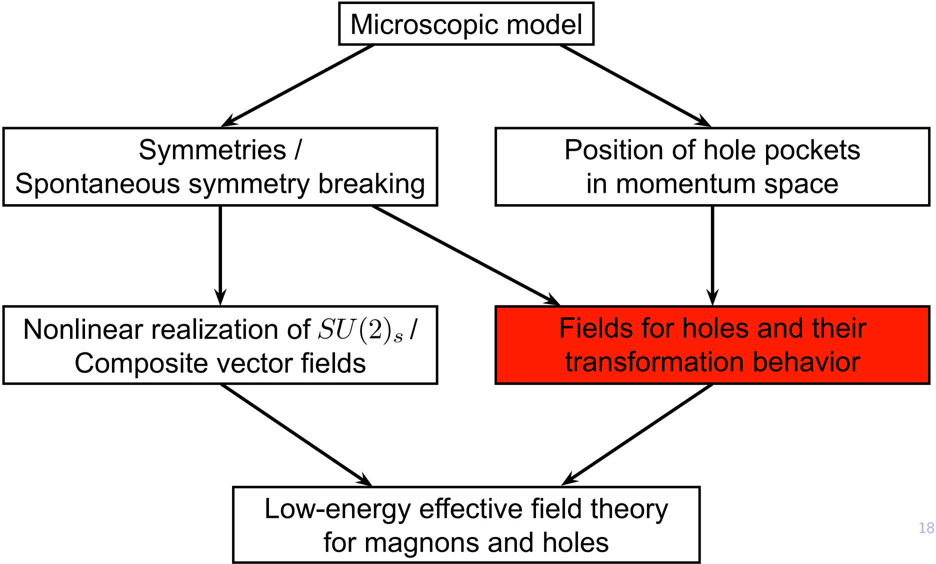
p_2 p_2

Hole pockets $\Longleftrightarrow$ Effective fields for holes



Symmetry considerations: Two half-pockets combine to a full pocket

Symmetry-based construction of effective theory



Transformation behavior of hole fields

- The symmetry properties of the underlying system have to be inherited by the effective theory!

Transformation behavior of hole fields

- The symmetry properties of the underlying system have to be inherited by the effective theory!
- Transformation rules for hole fields:

$$
\begin{aligned}
SU(2)_s: &\quad \psi^f_\pm(x)' = \exp(\pm i\alpha(x))\psi^f_\pm(x) \\
U(1)_Q: &\quad {}^Q\psi^f_\pm(x) = \exp(i\omega)\psi^f_\pm(x) \\
D_i: &\quad {}^{D_i}\psi^f_\pm(x) = \mp\exp(ik^f_i a)\exp(\mp i\varphi(x))\psi^f_\mp(x) \\
O: &\quad {}^O\psi^\alpha_\pm(x) = \mp\psi^\beta_\pm(Ox) \quad {}^O\psi^\beta_\pm(x) = \psi^\alpha_\pm(Ox) \\
R: &\quad {}^R\psi^\alpha_\pm(x) = \psi^\beta_\pm(Rx) \quad\quad {}^R\psi^\beta_\pm(x) = \psi^\alpha_\pm(Rx) \\
T: &\quad {}^T\psi^f_\pm(x) = \mp\exp(\mp i\varphi(Tx))\psi^{f\dagger}_\pm(Tx) \\
&\quad {}^T\psi^{f\dagger}_\pm(x) = \pm\exp(\pm i\varphi(Tx))\psi^f_\pm(Tx)
\end{aligned}
$$

Transformation behavior of hole fields

- The symmetry properties of the underlying system have to be inherited by the effective theory!
- Transformation rules for hole fields:

$$
\begin{aligned}
SU(2)_s: &\quad \psi^f_\pm(x)' = \exp(\pm i\alpha(x))\psi^f_\pm(x) \\
U(1)_Q: &\quad {}^Q\psi^f_\pm(x) = \exp(i\omega)\psi^f_\pm(x) \\
D_i: &\quad {}^{D_i}\psi^f_\pm(x) = \mp\exp(ik^f_i a)\exp(\mp i\varphi(x))\psi^f_\mp(x) \\
O: &\quad {}^O\psi^\alpha_\pm(x) = \mp\psi^\beta_\pm(Ox) \quad {}^O\psi^\beta_\pm(x) = \psi^\alpha_\pm(Ox) \\
R: &\quad {}^R\psi^\alpha_\pm(x) = \psi^\beta_\pm(Rx) \qquad {}^R\psi^\beta_\pm(x) = \psi^\alpha_\pm(Rx) \\
T: &\quad {}^T\psi^f_\pm(x) = \mp\exp(\mp i\varphi(Tx))\psi^{f\dagger}_\pm(Tx) \\
&\quad {}^T\psi^{f\dagger}_\pm(x) = \pm\exp(\pm i\varphi(Tx))\psi^f_\pm(Tx)
\end{aligned}
$$

- In a systematic derivative expansion: Construct the most general Lagrangian which respects all symmetries

Effective Lagrangian for magnons and holes

Lagrangian at leading order: Brügger et al. PRB 74 (2006) 224432

$$\begin{aligned}
\mathcal{L} = & \frac{\rho_s}{2}(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2}\partial_t \vec{e} \cdot \partial_t \vec{e}) \\
& + \sum_{\substack{f=\alpha,\beta \\ s=+,-}} \Big[M \psi_s^{f\dagger} \psi_s^f + \psi_s^{f\dagger} D_t \psi_s^f \\
& \quad + \frac{1}{2M'} D_i \psi_s^{f\dagger} D_i \psi_s^f + \sigma_f \frac{1}{2M''} \left(D_1 \psi_s^{f\dagger} D_2 \psi_s^f + D_2 \psi_s^{f\dagger} D_1 \psi_s^f \right) \\
& \quad + \Lambda \left(\psi_s^{f\dagger} v_1^s \psi_{-s}^f + \sigma_f \psi_s^{f\dagger} v_2^s \psi_{-s}^f \right) \\
& \quad + N_1 \psi_s^{f\dagger} v_i^s v_i^{-s} \psi_s^f + \sigma_f N_2 \left(\psi_s^{f\dagger} v_1^s v_2^{-s} \psi_s^f + \psi_s^{f\dagger} v_2^s v_1^{-s} \psi_s^f \right) \Big]
\end{aligned}$$

with

$$D_\mu \psi_\pm^f(x) = \left[\partial_\mu \pm i v_\mu^3(x)\right] \psi_\pm^f(x)$$

$$\sigma_\alpha = +1, \quad \sigma_\beta = -1$$

Effective Lagrangian for magnons and holes

Lagrangian at leading order: Brügger et al. PRB 74 (2006) 224432

$$\begin{aligned}\mathcal{L} = {} & \frac{\rho_s}{2}\left(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2}\partial_t \vec{e} \cdot \partial_t \vec{e}\right) \\ & + \sum_{\substack{f=\alpha,\beta \\ s=+,-}} \Big[M \psi_s^{f\dagger} \psi_s^f + \psi_s^{f\dagger} D_t \psi_s^f \\ & \quad + \frac{1}{2M'} D_i \psi_s^{f\dagger} D_i \psi_s^f + \sigma_f \frac{1}{2M''}\left(D_1 \psi_s^{f\dagger} D_2 \psi_s^f + D_2 \psi_s^{f\dagger} D_1 \psi_s^f\right) \\ & \quad + \Lambda\left(\psi_s^{f\dagger} v_1^s \psi_{-s}^f + \sigma_f \psi_s^{f\dagger} v_2^s \psi_{-s}^f\right) \\ & \quad + N_1 \psi_s^{f\dagger} v_i^s v_i^{-s} \psi_s^f + \sigma_f N_2\left(\psi_s^{f\dagger} v_1^s v_2^{-s} \psi_s^f + \psi_s^{f\dagger} v_2^s v_1^{-s} \psi_s^f\right)\Big]\end{aligned}$$

with

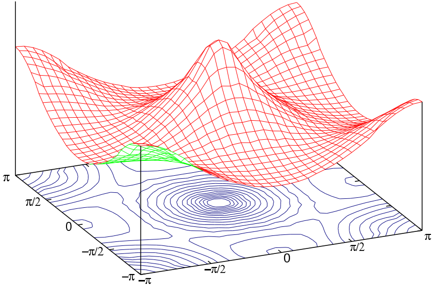
$$D_\mu \psi_\pm^f(x) = \left[\partial_\mu \pm i v_\mu^3(x)\right] \psi_\pm^f(x)$$

$$\sigma_\alpha = +1, \quad \sigma_\beta = -1$$

Outline

Electron pockets $\Longleftrightarrow$ Effective fields for electrons

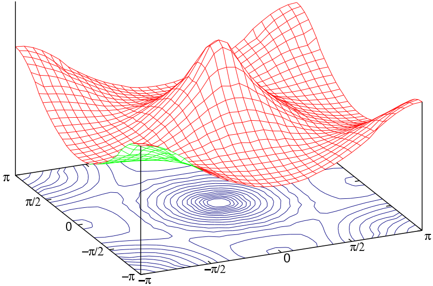
Single electron (away from half-filling) dispersion relation in the first Brillouin zone:



Minima at lattice momenta $\vec{k} = (\frac{\pi}{a}, 0)$ and $\vec{k} = (0, \frac{\pi}{a})$

Electron pockets $\Longleftrightarrow$ Effective fields for electrons

Single electron (away from half-filling) dispersion relation in the first Brillouin zone:



Minima at lattice momenta $\vec{k} = (\frac{\pi}{a}, 0)$ and $\vec{k} = (0, \frac{\pi}{a})$

Symmetry considerations: one single electron pocket!

$\Longrightarrow$ Fields for electrons: $\psi_{\pm}(x) \quad \psi^{\dagger}_{\pm}(x)$

Effective Lagrangian for magnons and electrons

Lagrangian at leading order:

$$\mathcal{L} = \frac{\rho_s}{2}(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2}\partial_t \vec{e} \cdot \partial_t \vec{e})$$
$$+ \sum_{s=+,-} \Big[M\psi_s^\dagger \psi_s + \psi_s^\dagger D_t \psi_s + \frac{1}{2M'} D_i \psi_s^\dagger D_i \psi_s + N\psi_s^\dagger v_i^s v_i^{-s} \psi_s$$
$$+ iK \left(D_1 \psi_s^\dagger v_1^s \psi_{-s} - \psi_s^\dagger v_1^s D_1 \psi_{-s} - D_2 \psi_s^\dagger v_2^s \psi_{-s} + \psi_s^\dagger v_2^s D_2 \psi_{-s} \right) \Big]$$

with

$$D_\mu \psi_\pm(x) = \left[\partial_\mu \pm i v_\mu^3(x)\right] \psi_\pm(x)$$

Brügger, Hofmann, Kämpfer, Moser, Pepe, and Wiese, PRB 75 (2007) 214405

Effective Lagrangian for magnons and electrons

Lagrangian at leading order:

$$\mathcal{L} = \frac{\rho_s}{2}(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2}\partial_t \vec{e} \cdot \partial_t \vec{e}) + \sum_{s=+,-} \Big[M \psi_s^\dagger \psi_s + \psi_s^\dagger D_t \psi_s + \frac{1}{2M'} D_i \psi_s^\dagger D_i \psi_s + N \psi_s^\dagger v_i^s v_i^{-s} \psi_s + iK \left(D_1 \psi_s^\dagger v_1^s \psi_{-s} - \psi_s^\dagger v_1^s D_1 \psi_{-s} - D_2 \psi_s^\dagger v_2^s \psi_{-s} + \psi_s^\dagger v_2^s D_2 \psi_{-s} \right) \Big]$$

with

$$D_\mu \psi_\pm(x) = \left[\partial_\mu \pm i v_\mu^3(x)\right] \psi_\pm(x)$$

Brügger, Hofmann, Kämpfer, Moser, Pepe, and Wiese, PRB 75 (2007) 214405

Effective Lagrangian for magnons and electrons

Lagrangian at leading order:

$$\mathcal{L} = \frac{\rho_s}{2}(\partial_i \vec{e} \cdot \partial_i \vec{e} + \frac{1}{c^2}\partial_t \vec{e} \cdot \partial_t \vec{e}) + \sum_{s=+,-} \Big[M\psi_s^\dagger \psi_s + \psi_s^\dagger D_t \psi_s + \frac{1}{2M'} D_i \psi_s^\dagger D_i \psi_s + N\psi_s^\dagger v_i^s v_i^{-s} \psi_s + iK\left(D_1 \psi_s^\dagger v_1^s \psi_{-s} - \psi_s^\dagger v_1^s D_1 \psi_{-s} - D_2 \psi_s^\dagger v_2^s \psi_{-s} + \psi_s^\dagger v_2^s D_2 \psi_{-s} \right) \Big]$$

with

$$D_\mu \psi_\pm(x) = \left[\partial_\mu \pm i v_\mu^3(x)\right] \psi_\pm(x)$$

Additional derivatives! $\Longrightarrow$ One-magnon exchange between electrons is weaker than between holes

Brügger, Hofmann, Kämpfer, Moser, Pepe, and Wiese, PRB 75 (2007) 214405

Comparison of Holes and Electrons

Holes:

- Carry flavor
- Elliptical pockets in BZ
- Interact with background at $O(p)$

Electrons:

- No flavor
- Circular pockets in BZ
- Interact with background at $O(p^2)$

Outline

Assumptions

To describe the antiferromagnet with finite doping, we assume

- Fermions are indoped homogeneously
- The magnetic background does not vary in time: $v_t = 0$
- Fermion contact interactions are small

- The homogeneous doping of fermions requires a homogeneous magnetic background.

  $\Rightarrow \quad v_i = const.$ up to a $U(1)_s$ "gauge" transformation:

$$
\begin{aligned}
v_i^3(x)' &= v_i^3(x) - \partial_i\alpha(x) = \sin^2\frac{\theta(x)}{2}\partial_i\varphi(x) - \partial_i\alpha(x) = c_i^3, \\
v_i^{\pm}(x)' &= v_i^{\pm}(x)\exp(\pm 2i\alpha(x)) \\
&= \frac{1}{2}\left[\sin\theta(x)\partial_i\varphi(x) \pm i\partial_i\theta(x)\right]\exp(\mp i(\varphi(x) - 2\alpha(x))) \\
&= c_i^{\pm}
\end{aligned}
$$

Theorem

The staggered magnetization $\vec{e}(x)$ configuration formed for uniform background fields c_i, c_i^3 is either homogeneous or a spiral

Brügger, Hofmann, Kämpfer, Pepe, and Wiese PRB 75 (2007) 014421

Hamiltonian Formulation I

Hole Hamiltonian:

$$H^f = \begin{pmatrix} M + \frac{(p_i - c_i^3)^2}{2M'} + \sigma_f \frac{(p_1 - c_1^3)(p_2 - c_2^3)}{M''} & \Lambda(c_1 + \sigma_f c_2) \\ \Lambda(c_1 + \sigma_f c_2) & M + \frac{(p_i + c_i^3)^2}{2M'} + \sigma_f \frac{(p_1 + c_1^3)(p_2 + c_2^3)}{M''} \end{pmatrix}$$

Electron Hamiltonian:

$$H = \begin{pmatrix} M + \frac{(p_i - c_i^3)^2}{2M'} + N c_i c_i & 2K(-p_1 c_1 + p_2 c_2) \\ 2K(-p_1 c_1 + p_2 c_2) & M + \frac{(p_i + c_i^3)^2}{2M'} + N c_i c_i \end{pmatrix}$$

Acting on the 2d spin space $\begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}$.

Hamiltonian Formulation II

Energy of the indoped fermions through diagonalization of the single-particle Hamiltonians

Hole Energy

$$E_{\pm}^{f}(\vec{p}) = M + \frac{p_i^2}{2M'} + \sigma_f \frac{p_1 p_2}{M''} \pm \Lambda|c_1 + \sigma_f c_2|$$

Electron Energy

$$E_{\pm}(\vec{p}) = M + \frac{p_i^2}{2M'} + N c_i c_i \pm 2K|p_1 c_1 - p_2 c_2|$$

- $\pm$ now refers to upper and lower energy states.
- Minimizing the energies of the fermions leads to $c_i^3 = 0$

Identification of Parameters

$c_i^3 = 0$ has important consequences. It can be shown, that

- $\theta(x) = \frac{\pi}{2}$
  $\Rightarrow$ The spiral plane lies in the plane CuO plane
- $\varphi(x) = 2c_i x_i$
- $k = 2\sqrt{c_1^2 + c_2^2} = const.$
  k is the spiral pitch

Background field contribution to the total energy density:

$$\epsilon = 2\rho_s v_i^+ v_i^- = 2\rho_s(c_1^2 + c_2^2)$$

Density of Indoped Holes - Total Energy Density

- Density per flavor:
  $n_\pm^f = \frac{1}{(2\pi)^2}\int_{P_\pm^f} d^2p = \frac{1}{2\pi} M_{\text{eff}} T_\pm^f, \quad M_{\text{eff}} = \frac{M' M''}{\sqrt{M''^2 - M'^2}}$
- Kinetic energy density:
  $t_\pm^f = \frac{1}{(2\pi)^2}\int_{P_\pm^f} d^2p \left(\frac{p_i^2}{2M'} + \sigma_f \frac{p_1 p_2}{M''}\right) = \frac{1}{4\pi} M_{\text{eff}} {T_\pm^f}^2$
- Total energy density:

$$\epsilon_h = \sum_{\substack{f=\alpha,\beta \\ s=+,-}} \left[(M + s\Lambda|c_1 + \sigma_f c_2|) n_s^f + t_s^f\right]$$

Minimize $\epsilon_h - \lambda n$: Minimizing the fermion energy density with fixed hole density. λ is a Lagrange multiplier.

Phases of Hole-Doped Antiferromagnets

4 Pocket Phase:

Fact

Bounded from below for $2\pi\rho_s > M_{eff}\Lambda^2$

$$c_i = 0 \qquad \epsilon_4 = \epsilon_0 + Mn + \frac{\pi n^2}{4M_{eff}}$$

3 Pocket Phase:

Fact

Bounded from below for $2\pi\rho_s > M_{eff}\Lambda^2$

$$|c_i| = \frac{\pi}{2}\frac{\Lambda n}{6\pi\rho_s - M_{eff}\Lambda^2} \qquad \epsilon_3 = \epsilon_0 + Mn + \frac{\pi}{3M_{eff}}\left(1 - \frac{1}{2}\frac{M_{eff}\Lambda^2}{6\pi\rho_s - M_{eff}\Lambda^2}\right)n^2$$

Phases of Hole-Doped Antiferromagnets

2 Pocket Phase:

Fact

Bounded from below for $2\pi\rho_s > \frac{1}{2} M_{eff}\Lambda^2$

$$|c_{1,2}| = \frac{\Lambda}{4\rho_s} n, \ |c_{2,1}| = 0, \quad \epsilon_2 = \epsilon_0 + Mn + \left(\frac{\pi}{2M_{eff}} - \frac{\Lambda^2}{8\rho_s}\right) n^2.$$

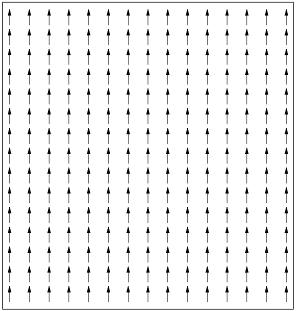
1 Pocket Phase:

Fact

Always bounded from below but unstable against forming inhomogeneities for $2\pi\rho_s < \frac{1}{2} M_{eff}\Lambda^2$

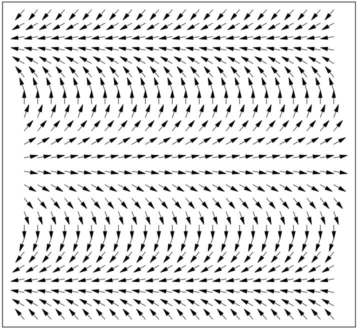
$$|c_1| = |c_2| = \frac{\Lambda}{4\rho_s} n \quad \epsilon_1 = \epsilon_0 + Mn + \left(\frac{\pi}{M_{eff}} - \frac{\Lambda^2}{4\rho_s}\right) n^2$$

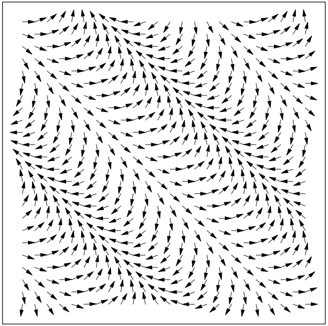
Phases of Hole-Doped Antiferromagnets



Homogeneous Phase
4 Hole Pockets

Phases of Hole-Doped Antiferromagnets

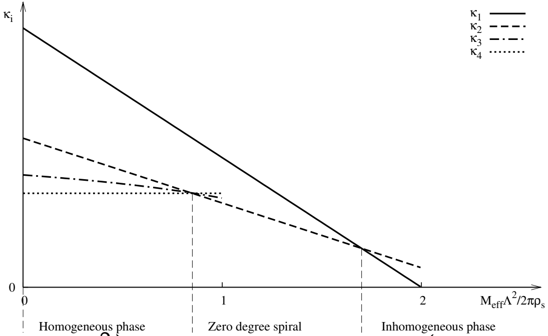




0 Degree Spiral
2 Hole Pockets

45 Degrees Spiral
3 (or 1) Hole Pockets

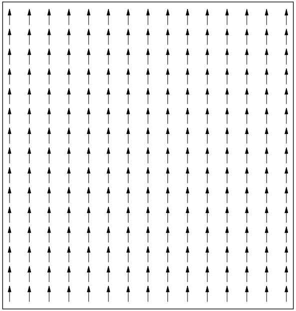
Stability of Phases for Hole Doping



$$\kappa_1 = \left(\frac{\pi}{M_{\text{eff}}} - \frac{\Lambda^2}{\rho_s}\right) \qquad \kappa_3 = \frac{\pi}{3M_{\text{eff}}}\left(1 - \frac{1}{2}\frac{M_{\text{eff}}\Lambda^2}{6\pi\rho_s - M_{\text{eff}}\Lambda^2}\right)$$

$$\kappa_2 = \left(\frac{\pi}{2M_{\text{eff}}} - \frac{\Lambda^2}{8\rho_s}\right) \qquad \kappa_4 = \frac{\pi}{4M_{\text{eff}}}$$

Homogeneous Phase for Electron-Doped Antiferromagnets



Homogeneous Phase
No Spiral Phases

Outline

Conclusions

- High-T superconductors: Condensed matter analog of baryon chiral perturbation theory
- We have constructed a systematic low-energy effective field theory for lightly doped antiferromagnets
- Using the effective theory we have investigated spiral phases in hole- and electron-doped cuprates
- While spiral phases do exist for hole-doping, they are absent in electron-doped cuprates
- We also calculated the one-magnon-exchange potential and investigated the possibility of hole-hole and electron-electron bound states

Outlook

- Analysis of materials with other lattice geometries: Honeycomb and Triangular lattices
- Incorporation of Phonons as low-energy degrees of freedom
- Systematic treatment of loop graphs
- Towards the mysterious Mechanism of high-T superconductivity

Nonlinear Realization of $SU(2)_s$ on the fermions

$$u(x)' = h(x)u(x)g^\dagger \quad C_x' = gC_x$$

Nonlinear Realization of $SU(2)_s$ on the fermions

$$u(x)' = h(x)u(x)g^\dagger \quad C_x' = gC_x$$

$$\Psi^X(x)' = h(x)u(x)C_x = h(x)\Psi^X(x)$$

$$\Psi^X(x) = \begin{pmatrix} \psi_+^X(x) & \psi_-^{X\dagger}(x) \\ \psi_-^X(x) & -\psi_+^{X\dagger}(x) \end{pmatrix}, \quad x \in \textit{even sublattice}$$

$$\Psi^X(x) = \begin{pmatrix} \psi_+^X(x) & -\psi_-^{X\dagger}(x) \\ \psi_-^X(x) & \psi_+^{X\dagger}(x) \end{pmatrix}, \quad x \in \textit{odd sublattice.}$$

The global spin rotation symmetry is also realized locally on the fermions.

Sublattice Structure 1

$$
\begin{array}{ccccccccc}
 & & | & & | & & | & & | \\
— & B & — & A & — & B & — & A & — \\
 & | & & | & & | & & | & \\
— & A & — & B & — & A & — & B & — \\
 & | & & | & & | & & | & \\
— & B & — & A & — & B & — & A & — \\
 & | & & | & & | & & | & \\
— & A & — & B & — & A & — & B & — \\
 & | & & | & & | & & | &
\end{array}
$$

$$k = (k1, k2) \in \left\{(0,0); \left(\frac{\pi}{a}, \frac{\pi}{a}\right)\right\}$$

Sublattice Structure 2

$$k = (k_1, k_2) \in \left\{ (0,0)\,, \left(\frac{pi}{a}, \frac{\pi}{a}\right)\,, \left(\frac{\pi}{a}, 0\right)\,, \left(0, \frac{\pi}{a}\right)\,, \left(\pm\frac{\pi}{2a}, \pm\frac{\pi}{2a}\right) \right\}$$

$$\begin{array}{ccccccccc}
 & | & & | & & | & & | & \\
— & G & — & H & — & E & — & F & — \\
 & | & & | & & | & & | & \\
— & C & — & D & — & A & — & B & — \\
 & | & & | & & | & & | & \\
— & E & — & F & — & G & — & H & — \\
 & | & & | & & | & & | & \\
— & A & — & B & — & C & — & D & — \\
 & | & & | & & | & & | &
\end{array}$$

From Charge Carriers to Grassmann numbers

From Charge Carriers to Grassmann numbers

Hubbard model in manifestly $SU(2)_{\vec{Q}}$ invariant form (at half filling)

$$\mathcal{H} = -\frac{t}{2}\sum_{x,i}\mathrm{Tr}[C_x^\dagger C_{x+\hat{i}} + C_{x+\hat{i}}^\dagger C_x] + \frac{U}{12}\sum_x \mathrm{Tr}[C_x^\dagger C_x C_x^\dagger C_x].$$

$$C_x = \begin{pmatrix} c_{x\uparrow} & (-1)^x c_{x\downarrow}^\dagger \\ c_{x\downarrow} & -(-1)^x c_{x\uparrow}^\dagger \end{pmatrix}$$

From Charge Carriers to Grassmann numbers

Hubbard model in manifestly $SU(2)_{\vec{Q}}$ invariant form (at half filling)

$$\mathcal{H} = -\frac{t}{2} \sum_{x,i} \mathrm{Tr}[C_x^\dagger C_{x+\hat{i}} + C_{x+\hat{i}}^\dagger C_x] + \frac{U}{12} \sum_x \mathrm{Tr}[C_x^\dagger C_x C_x^\dagger C_x].$$

$$C_x = \begin{pmatrix} c_{x\uparrow} & (-1)^x c_{x\downarrow}^\dagger \\ c_{x\downarrow} & -(-1)^x c_{x\uparrow}^\dagger \end{pmatrix}$$

$$\Psi^X(x) = u(x) C_x \quad X := sublatticeindex$$

Solution for $\vec{e}(x)$ for constant background fields

$$\vec{e}(x) = \left(sin\theta(x)cos\varphi(x), \sin\theta(x)sin\varphi(x), \cos\theta(x)\right)$$

$$\cos\theta(x) = \frac{1}{\sqrt{1+\left(\frac{c_i}{c_i^3}\right)^2}}\left[\cos\eta + \frac{c_i}{c_i^3}\sin\eta\cos\left(2\sqrt{1+\left(\frac{c_i}{c_i^3}\right)^2}c_i^3 x_i\right)\right].$$

$$\varphi(x) = \mathrm{atan}\left(\frac{\frac{c_i}{c_i^3}\sin\left(2\sqrt{1+\left(\frac{c_i}{c_i^3}\right)^2}c_i^3 x_i\right)}{\sin\eta - \frac{c_i}{c_i^3}\cos\eta\cos\left(2\sqrt{1+\left(\frac{c_i}{c_i^3}\right)^2}c_i^3 x_i\right)}\right).$$

Antiferromagnetism: Hubbard model

Four possible states at each lattice site: $|0\rangle, \quad |\uparrow\rangle, \quad |\downarrow\rangle, \quad |\uparrow\downarrow\rangle$

Antiferromagnetism: Hubbard model

Four possible states at each lattice site: $|0\rangle, \quad |\uparrow\rangle, \quad |\downarrow\rangle, \quad |\uparrow\downarrow\rangle$

Assumptions:

- Half-filling: in average one fermion per lattice site
- Strong coupling limit: $U \gg t$

Antiferromagnetism: Hubbard model

Four possible states at each lattice site: $|0\rangle, \quad |\uparrow\rangle, \quad |\downarrow\rangle, \quad |\uparrow\downarrow\rangle$

Assumptions:

- Half-filling: in average one fermion per lattice site
- Strong coupling limit: $U \gg t$

Consequences:

- $|\uparrow\downarrow\rangle$ huge energy cost $\Longrightarrow$ Ground state consists of $|\uparrow\rangle, |\downarrow\rangle$
- Enormous degeneracy of states

Antiferromagnetism: Hubbard model

Four possible states at each lattice site: $|0\rangle, \quad |\uparrow\rangle, \quad |\downarrow\rangle, \quad |\uparrow\downarrow\rangle$

Assumptions:

- Half-filling: in average one fermion per lattice site
- Strong coupling limit: $U \gg t$

Consequences:

- $|\uparrow\downarrow\rangle$ huge energy cost $\Longrightarrow$ Ground state consists of $|\uparrow\rangle, |\downarrow\rangle$
- Enormous degeneracy of states

How does the ground state order?

- By hopping system can lower its energy
- Hopping only possible for antiparallel spins

$\Longrightarrow$ Antiferromagnetic spin alignment is favored!

Relating microscopic operators to effective fields I

With the matrix-valued operator

$$C_x = \begin{pmatrix} c_{x\uparrow} & (-1)^x \; c^\dagger_{x\downarrow} \\ c_{x\downarrow} & -(-1)^x c^\dagger_{x\uparrow} \end{pmatrix}$$

the Hubbard Hamiltonian can be written

$$H = -\frac{t}{2} \sum_{x,i} \mathrm{Tr}[C_x^\dagger C_{x+\hat{i}} + C^\dagger_{x+\hat{i}} C_x] + \frac{U}{12} \sum_x \mathrm{Tr}[C_x^\dagger C_x C_x^\dagger C_x]$$
$$- \frac{\mu}{2} \sum_x \mathrm{Tr}[C_x^\dagger C_x \sigma_3]$$

Relating microscopic operators to effective fields II

- Defining new lattice operators with the help of the diagonalizing matrix $u(x)$:

$$\Psi_x^{A,B,\dots,H} = u(x) C_x, \qquad x \in A, B, \dots, H$$

Relating microscopic operators to effective fields II

- Defining new lattice operators with the help of the diagonalizing matrix $u(x)$:

$$\Psi_x^{A,B,\ldots,H} = u(x) C_x, \qquad x \in A, B, ..., H$$

- Work out symmetry transformation properties

Relating microscopic operators to effective fields II

- Defining new lattice operators with the help of the diagonalizing matrix $u(x)$:

$$\Psi_x^{A,B,...,H} = u(x) C_x, \qquad x \in A, B, ..., H$$

- Work out symmetry transformation properties
- Replace lattice operators by effective Grassmann fields

$$\Psi_x^{A,B,...,H} \quad \longrightarrow \quad \Psi^{A,B,...,H}(x)$$

Relating microscopic operators to effective fields II

- Defining new lattice operators with the help of the diagonalizing matrix $u(x)$:

$$\Psi_x^{A,B,\dots,H} = u(x) C_x, \qquad x \in A, B, \dots, H$$

- Work out symmetry transformation properties
- Replace lattice operators by effective Grassmann fields

$$\Psi_x^{A,B,\dots,H} \quad \longrightarrow \quad \Psi^{A,B,\dots,H}(x)$$

- Postulate: Transformation properties are inherited!

Accidental Galilean boost invariance

$$G: \qquad {}^G P(x) = P(Gx), \qquad Gx = (\vec{x} - \vec{v}\ t, t),$$
$$ {}^G\psi^f_\pm(x) = \exp\left(\vec{p}^f \cdot \vec{x} - \omega^f t\right) \psi^f_\pm(Gx),$$
$$ {}^G\psi^{f\dagger}_\pm(x) = \psi^{f\dagger}_\pm(Gx) \exp\left(-\vec{p}^f \cdot \vec{x} + \omega^f t\right),$$

with $\vec{p}^f = (p^f_1, p^f_2)$ and ω^f given by

$$p^f_1 = \frac{M'}{1-(M'/M'')^2}\left[v_1 - \sigma_f \frac{M'}{M''} v_2\right],$$
$$p^f_2 = \frac{M'}{1-(M'/M'')^2}\left[v_2 - \sigma_f \frac{M'}{M''} v_1\right],$$
$$\omega^f = \frac{{p^f_i}^2}{2M'} + \sigma_f \frac{p^f_1 p^f_2}{M''} = \frac{M'}{1-(M'/M'')^2}\left[\frac{1}{2}(v_1^2 + v_2^2) - \sigma_f \frac{M'}{M''} v_1 v_2\right]$$

Transformation behavior of electron fields

$$
\begin{aligned}
SU(2)_s: &\quad \psi_\pm(x)' = \exp(\pm i\alpha(x))\psi_\pm(x) \\
U(1)_Q: &\quad {}^Q\psi_\pm(x) = \exp(i\omega)\psi_\pm(x) \\
D_i: &\quad {}^{D_i}\psi_\pm(x) = \mp\exp(ik_i a)\exp(\mp i\varphi(x))\psi_\mp(x) \\
O: &\quad {}^O\psi_\pm(x) = \pm\psi_\pm(Ox) \\
R: &\quad {}^R\psi_\pm(x) = \psi_\pm(Rx) \\
T: &\quad {}^T\psi_\pm(x) = \exp(\mp i\varphi(Tx))\psi^\dagger_\pm(Tx) \\
&\quad {}^T\psi^\dagger_\pm(x) = -\exp(\pm i\varphi(Tx))\psi_\pm(Tx)
\end{aligned}
$$