Lattice Study of the Conformal Window in QCD-Like Theories

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PRL 100, 171607 (2008)
Longer Paper Soon

XIII Mexican School of
Particles and Fields

Beyond the Standard Model

Conformal or Near-Conformal Behavior in the IR:

Dynamical Electroweak Symmetry Breaking. (Walking Technicolor)

New Conformal Sector?

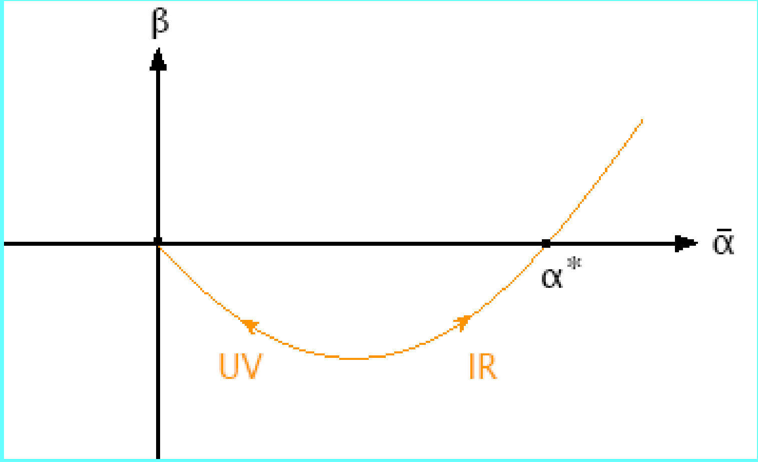
SUSY Flavor Hierarchies (Nelson & Strassler 2000/01)

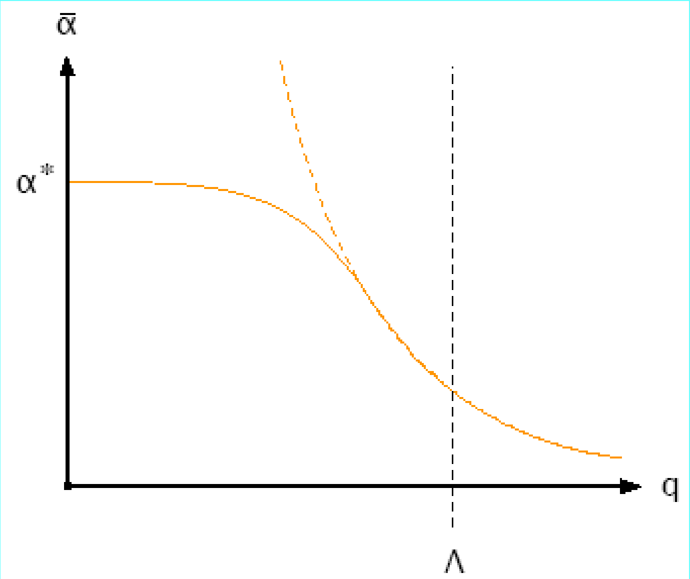
For an asymptotically free theory, an IR fixed point can emerge already in the two-loop β function, depending on the number of fermions N_f

Gross and Wilczek, antiquity
Caswell, 1974
Banks and Zaks, 1982
Many Others

Reliable if the number of fermions is very close to the number at which asymptotic freedom is lost

Cartoons





α* increases as N_f decreases.

Should be a range of N_f where IR fixed point exists, not necessarily accessible in PT. (This is known in certain SUSY theories.)

Possibilities

(1) $\alpha^* < \alpha_c^*$ $(N_f > N_{fc})$

Conformal IR behavior (Non-abelian coulomb phase).

(2) $\alpha^* > \alpha_c^*$ $(N_f < N_{fc})$

Chiral symmetry breaking, confinement

(3) $\alpha^* \gtrsim \alpha_c^*$ $(N_f \lesssim N_{fc})$ (fine tuning?)

If the transition is continuous, breaking scale $<< \Lambda$,

$\Rightarrow$ Walking at intermediate scales.

Questions

1. Value of N_{fc}?

2. Order of the phase transition?

3. Physical states below and near the transition?

4. Implications for EW precision studies? (The S parameter etc)?

5. Implications for the LHC?

N_{fc} in SU(N) QCD

- Degree-of-Freedom Inequality (Cohen, Schmaltz, TA 1999). Fundamental rep:

  $N_{fc} \leq \;\; 4\,N[1 - 1/18N^2 + \ldots]$

- Gap-Equation Studies, Instantons (1996): $N_{fc} \cong 4\,N$

- Lattice Simulation (Iwasaki et al, Phys Rev D69, 014507 2004):

  $6 < N_{fc} < 7 \qquad$ For $N = 3$

N_{fc} in SUSY SU(N) QCD

Degree of Freedom Inequality:

$$N_{fc} \leq (3/2)\, N$$

Seiberg Duality: $N_{fc} = (3/2)\, N$!!

Weakly coupled magnetic dual in the vicinity of this value

Some Quasi-Perturbative Studies of the Conformal Window in QCD-like Theories

1. Gap – Equation studies in the mid 1990s

2. V. Miransky and K. Yamawaki hep-th/9611142 (1996)

3. E. Gardi, G. Grunberg, M. Karliner hep-ph/9806462 (1998)

4. E. Gardi and G. Grunberg JHEP/004A/1298 (2004)

   "The IRFP is perturbative in the entire conformal window"

5. Kurachi and Shrock, hep-ph/0605290

6. H. Terao and A. Tsuchiya arXiv:0704.3659 [hep-ph] (2007)

Lattice-Simulation Study of the Extent of the Conformal window in an SU(3) Gauge Theory with Dirac Fermions in the Fundamental Representation

Previous Lattice Work with Many Light Fermions

1. Brown et al (Columbia group) Phys. Rev. D12, 5655 (1992)
   $N_f = 8$
2. Damgaard, Heller, Krasnitz and Oleson, hep-lat/9701008
   $N_f = 16$
3. R. Mahwinney, hep/lat/9701030(1) $\left(N_f \to 4\right)$,
   Nucl.Phys.Proc.Suppl.83:57-66,2000. e-Print: hep-lat/0001032
4. C. Sui, Flavor dependence of quantum chromodynamics. PhD thesis, Columbia University, New York, NY, 2001. UMI-99-98219
5. Iwasaki et al, Phys. Rev, D69, 014507 (2004)

Focus: Gauge Invariant and Non-Perturbative Definition of the Running Coupling Deriving from the Schroedinger Functional of the Gauge Theory

ALPHA Collaboration: Luscher, Sommer, Weisz, Wolff, Bode, Heitger, Simma, …

Using Staggered Fermions as in

U. Heller, Nucl. Phys. B504, 435 (1997)
Miyazaki & Kikukawa

$O(a^2)$ Chiral Breaking $\Longrightarrow$ Remaining Continuous Chiral Symmetry

Focus on N_f = Multiples of 4:

16: Perturbative IRFP

$$\boxed{\begin{array}{ll} 12: & \text{IRFP “expected”, Simulate} \\ 8: & \text{IRFP uncertain , Simulate} \end{array}}$$

4 : Confinement, ChSB

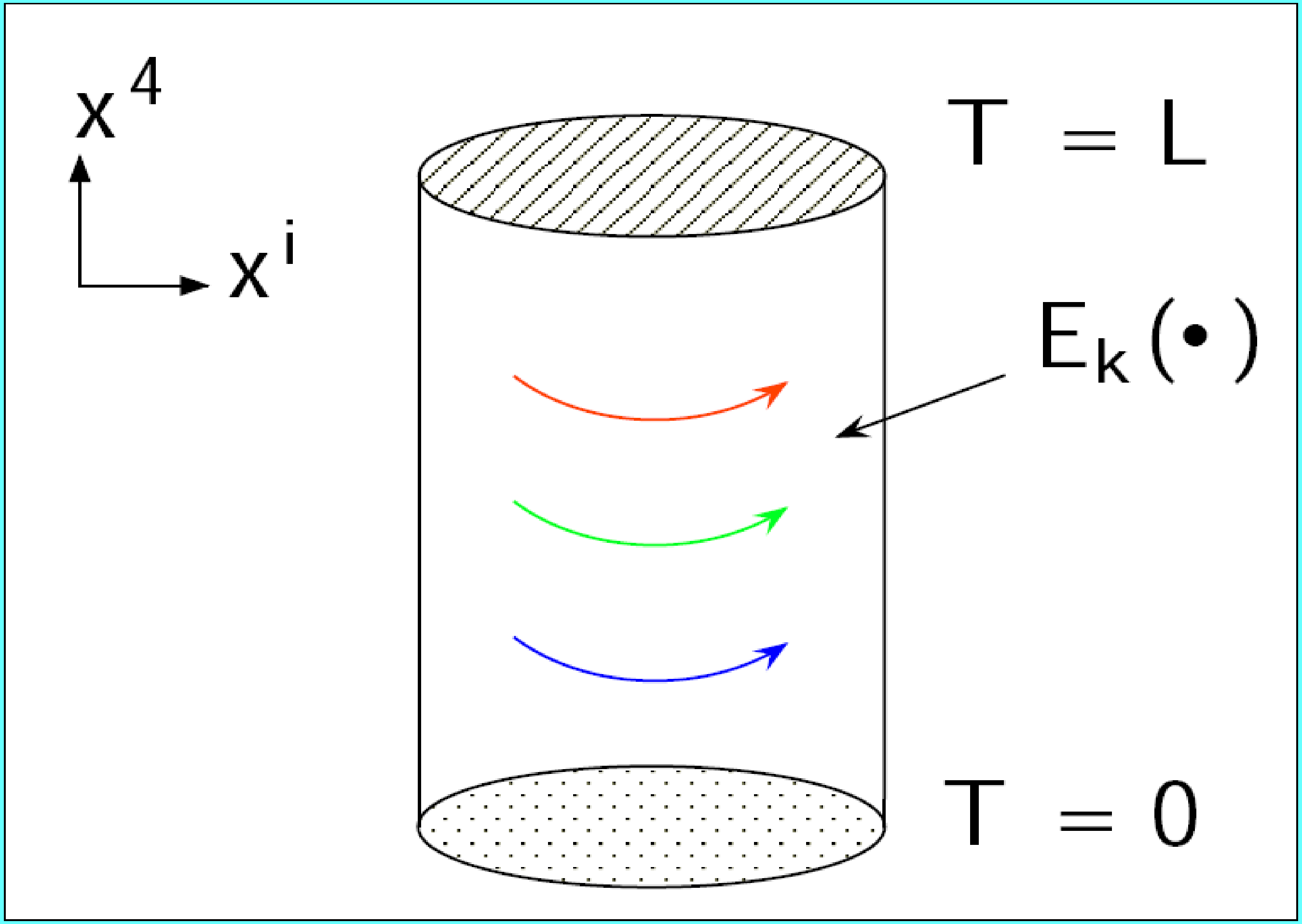
The Shroedinger Functional

- Transition amplitude from a prescribed state at t=0 to one at t=T (Dirichlet BC).
- Euclidean path integral with Dirichlet BC in time and periodic in space (L) to describe a constant chromo-electric background field.

$$Z\left[W,\zeta,\bar{\zeta};W',\zeta',\bar{\zeta}'\right]=$$

$$\int\left[DUD\chi D\bar{\chi}\right]e^{-S_G(W,W')-S_F(W,W',\zeta,\bar{\zeta},\zeta',\bar{\zeta}')}$$

Picture



Abelian Boundary Fields

$$W_k(x) = diag\left(e^{i\phi_1/L}, e^{i\phi_2/L}, e^{i\phi_3/L}\right),$$
$$W_k'(x) = diag\left(e^{i\phi_1/L}, e^{i\phi_2/L}, e^{i\phi_3/L}\right).$$

$$\phi_1 = -\frac{\pi}{3} + \eta, \quad \phi_2 = -\tfrac{1}{2}\eta, \quad \phi_3 = -\frac{\pi}{3} + \tfrac{1}{2}\eta,$$

$$\phi_1' = -\pi - \eta, \quad \phi_2' = \frac{\pi}{3} + \tfrac{1}{2}\eta, \quad \phi_3' = \frac{2\pi}{3} + \tfrac{1}{2}\eta.$$

- Constant chromoelectric background field of strength $\frac{1}{L}$
- Can set $m_f = 0$

Schroedinger Functional (SF) Running Coupling on Lattice

Define: $$\frac{1}{\bar{g}^2(L,T)} \equiv \frac{-1}{k}\frac{\partial}{\partial \eta}\log Z\Big|_{\eta=0},$$

$$= \frac{1}{g_0^{\,2}} + 0(1) + 0\left(g_0^{\,2}\right) + \ldots$$

Response of system to small changes in the background field.

$$k = 12\left(\frac{L}{a}\right)^2\left[\sin\left(\frac{2\pi a^2}{3LT}\right) + \sin\left(\frac{\pi a^2}{3LT}\right)\right]$$

SF Running Coupling

Then, to remove the O(a) bulk lattice artifact

$$\frac{1}{\bar{g}^2(L)} = \frac{1}{2}\left[\frac{1}{\bar{g}^2(L, L-a)} + \frac{1}{\bar{g}^2(L, L+a)}\right]$$

Depends on only one scale L

Look for conformal symmetry (IRFP) <u>at</u> the box scale L

Loop Expansion

$$L\frac{\partial}{\partial L}\bar{g}^2(L) = \beta\left(\bar{g}^2(L)\right) = b_1\bar{g}^4(L) + b_2\bar{g}^6(L) + b_3\bar{g}^8(L) + \ldots.$$

$$b_1 = \frac{2}{(4\pi)^2}\left(11 - \frac{2}{3}N_f\right), \qquad b_2 = \frac{2}{(4\pi)^4}\left(102 - \frac{38}{3}N_f\right)$$

$$b_3 = b_3^{\overline{MS}} + \frac{b_2 c_2}{2\pi^2} - \frac{b_1(c_3 - c_2)}{8\pi^2}$$

$$b_3^{\overline{MS}} = \frac{1}{(4\pi)^6}\left[\frac{2857}{2} - \frac{5033}{18}N_f + \frac{325}{54}N_f^2\right]$$

$$c_2 = 1.256 + 0.04N_f$$

$$c_3 = c_2^2 + 1.20 + 0.14N_f - 0.03N_f^2$$

Loop Expansion

$N_f = 16$ IRFP at $g{*}_{SF}^{2} = 0.47$ $\left(\frac{\bar{g}^2}{4\pi^2} \approx .01\right)$

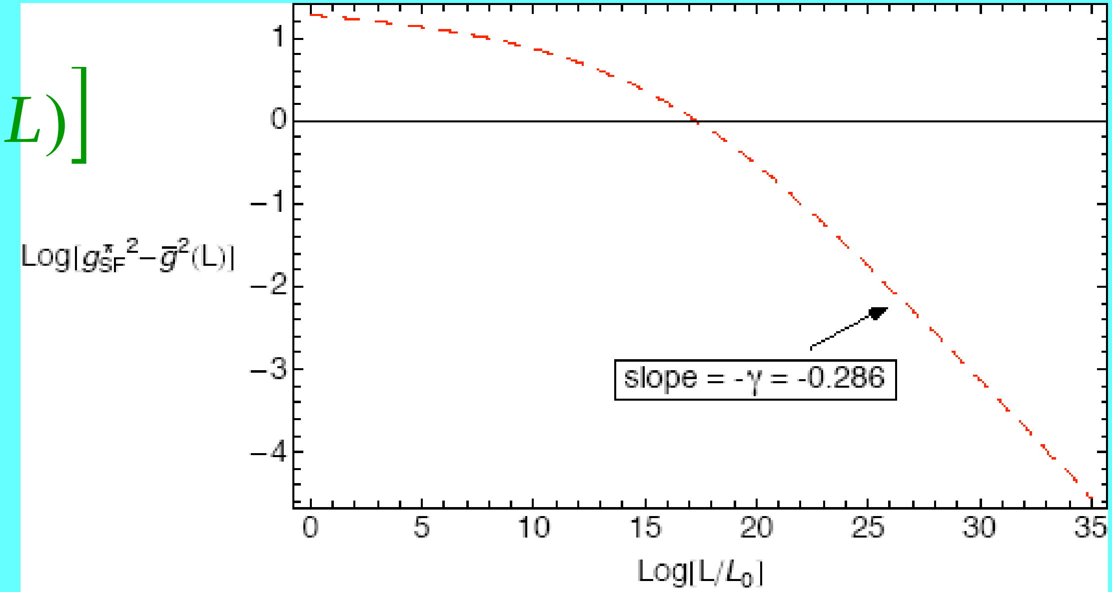
$N_f = 12$ IRFP at $g{*}_{SF}^{2} = 5.18$ $\left(\frac{\bar{g}^2}{4\pi^2} \approx .13\right)$

$N_f \leq 8$ No perturbative IRFP

Loop Expansion

Linearize β near the $N_f = 12$ IRFP

$$\beta(\bar{g}^2(L)) \cong \gamma\left[g_{SF}^{*2} - \bar{g}^2(L)\right]$$



Then:

$$\bar{g}^2(L) \xrightarrow{L\to\infty} g_{SF}^{*2} - \frac{const}{L^{\gamma}}$$

Lattice Simulations

MILC Code (Heller)

Staggered Fermions

$$N_f = 8, 12$$

Range of Lattice Couplings g_0^2 (= $6/\beta$) and Lattice Sizes $L/a \rightarrow 20$

O(a) Lattice Artifacts due to Dirichlet Boundary Conditions

Statistical and Systematic Error

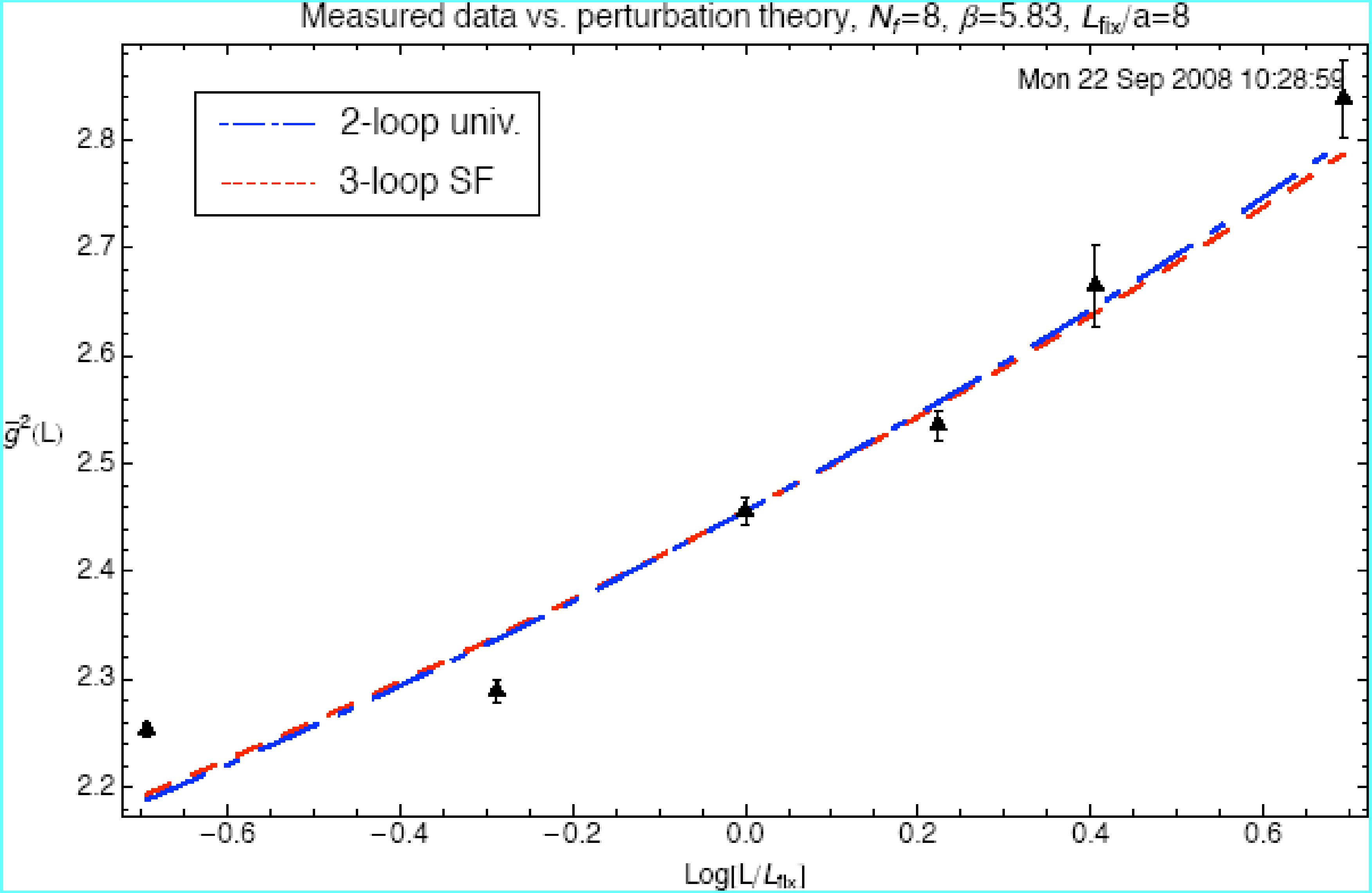
1. Numerical-simulation error
2. Interpolating-function error
3. Continuum-extrapolation error

Statistics Dominates

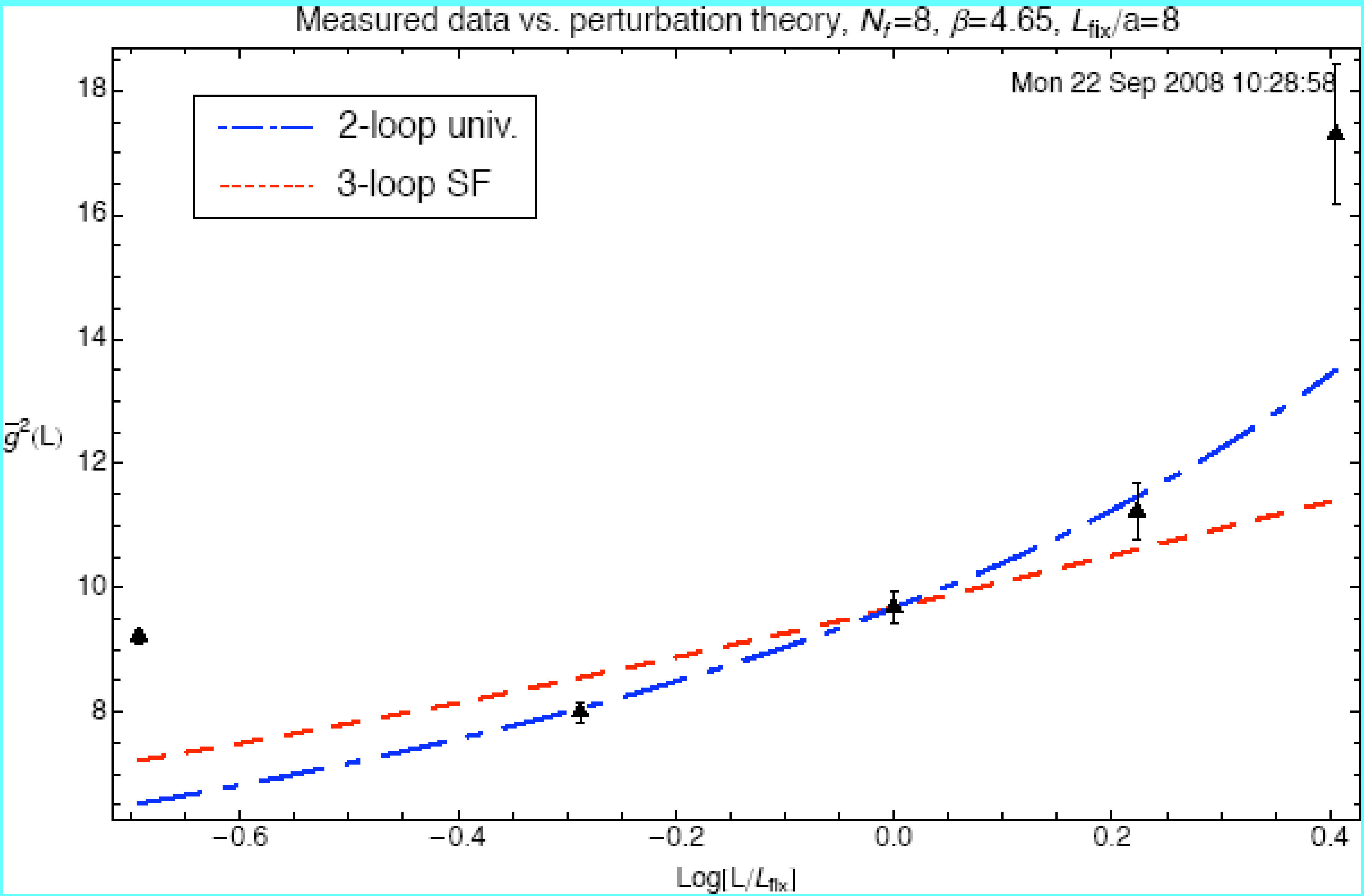
$N_f = 8$ Data

Blue (Red) numbers indicate the number of trajectories collected for • =• + 1 (• • 1).

•²(•)	• • •													
•	4	• traj	6	• traj	8	• traj	10	• traj	12	• traj	16	• traj	20	• traj
4.200	027857(360000)	91• 162•												
4.300	-816(130)	77• 87•												
4.400	161(45)	60• 86•												
4.450	41.5(1.6)	82• 82•	40.5(3.2)	33• 75•										
4.500	23.79(61)	82• 82•	17.60(53)	82• 82•	34.9(3.4)	26• 30•								
4.550	15.94(32)	82• 82•	12.27(29)	82• 82•	18.1(1.0)	61• 59•								
4.600	11.63(13)	47• 47•	9.70(28)	82• 82•	11.1(1.9)	47• 53•	14.52(68)	81• 81•	16.6(1.8)	16• 14•				
4.650	9.22(11)	82• 82•	8.00(16)	82• 82•	9.61(23)	81• 81•	11.22(45)	81• 81•	17.3(1.1)	64• 77•				
4.700	7.55(15)	42• 42•	6.79(19)	42• 42•	8.58(56)	26• 31•	11.15(81)	41• 41•						
4.800	5.86(12)	42• 42•	5.64(13)	42• 42•	6.60(23)	33• 40•	7.19(25)	41• 41•	7.48(41)	24• 25•	9.8(1.4)	12• 12•		
4.900	4.990(80)	42• 42•	5.03(24)	42• 42•	5.31(17)	37• 42•								
5.000	4.169(63)	42• 42•	4.159(97)	40• 40•	4.76(12)	29• 30•	4.99(16)	41• 41•	5.76(26)	25• 25•	6.10(61)	37• 38•		
5.100	3.755(34)	41• 41•	3.803(59)	41• 41•	4.226(96)	41• 41•								
5.200	3.382(42)	41• 41•	3.385(26)	41• 41•	3.771(74)	41• 35•								
5.300	3.115(26)	41• 41•	3.099(26)	41• 41•	3.339(74)	53• 54•								
5.400	2.882(29)	41• 41•	3.022(41)	41• 41•	3.171(39)	41• 41•								
5.500	2.743(26)	41• 41•	2.736(17)	40• 40•	2.980(34)	24• 24•	3.113(52)	41• 41•	3.4(0.1)	40• 40•	3.355(82)	30• 31•		
5.600	2.578(21)	41• 41•	2.599(21)	41• 41•	2.794(31)	41• 41•								
5.700	2.4058(83)	41• 41•	2.458(18)	41• 41•	2.591(14)	41• 41•								
5.800	2.2859(80)	37• 41•	2.324(11)	41• 41•	2.494(16)	41• 41•								
5.830	2.2531(71)	41• 41•	2.2889(99)	41• 41•	2.456(13)	41• 41•	2.536(14)	41• 41•	2.665(39)	41• 41•	2.839(36)	31• 32•		
5.900	2.211(14)	41• 41•	2.2244(88)	41• 41•	2.344(14)	41• 41•								



Measured data vs. perturbation theory, N_f=8, β=5.83, L_{fix}/a=8
Mon 22 Sep 2008 10:28:59
2-loop univ.
3-loop SF
$\bar{g}^2$(L)
2.2
2.3
2.4
2.5
2.6
2.7
2.8
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
Log[L/L_{fix}]



Measured data vs. perturbation theory, N_f=8, β=4.65, L_{fix}/a=8
Mon 22 Sep 2008 10:28:58
2-loop univ.
3-loop SF
$\bar{g}^2$(L)
18
16
14
12
10
8
−0.6
−0.4
−0.2
0.0
0.2
0.4
Log[L/L_{fix}]

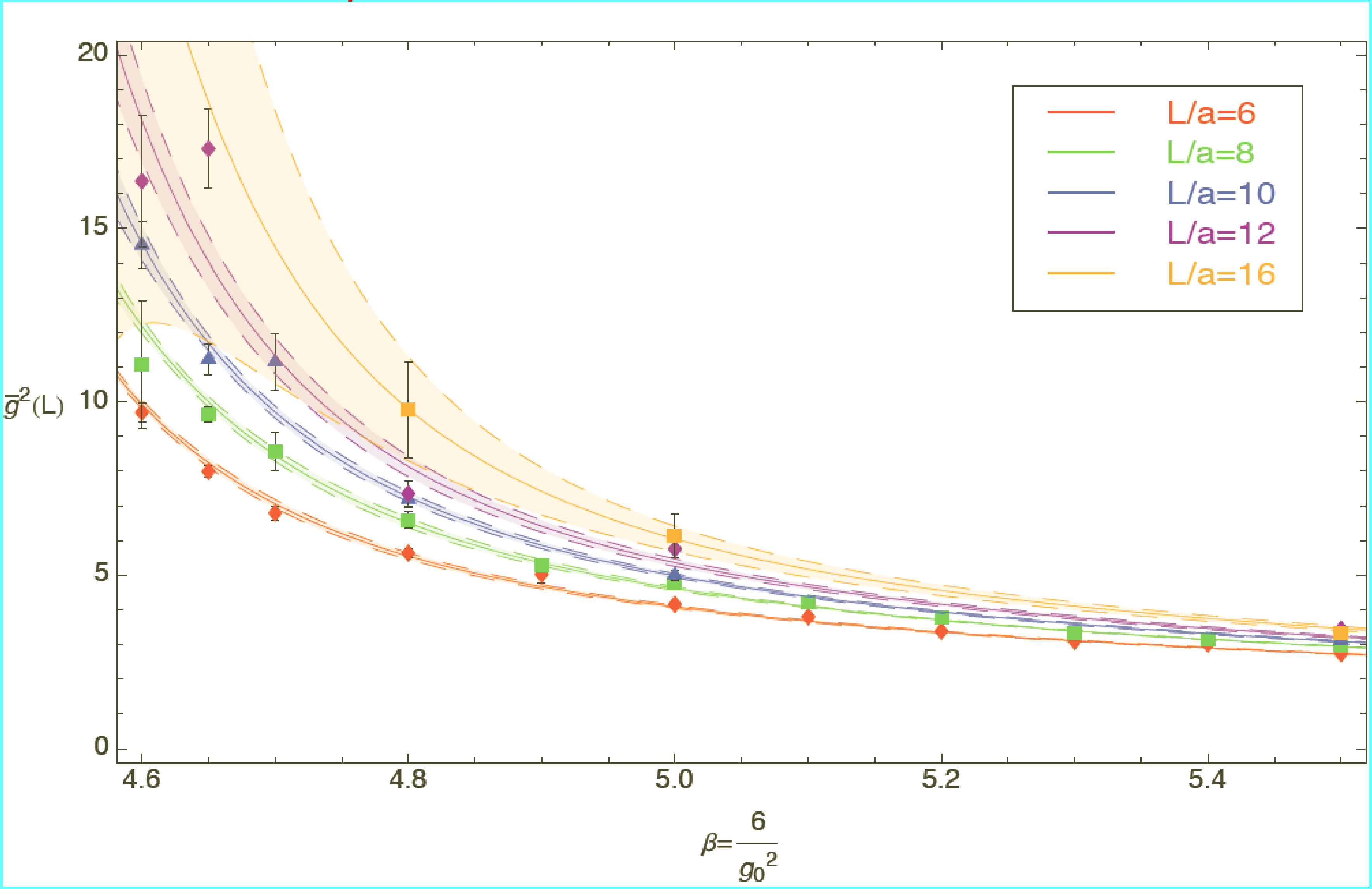
$$\overline{g}^2(\beta, L/a) = \sum_{i=1}^{4} \frac{c_i}{(\beta - \beta_0)^i} + a_0 e^{-m_0 \beta}$$

$$c_i = (2N_c)^i \left(c_{i0} + c_{i1}(a/L) + c_{i2}(a/L)^2 \right)$$

$$\beta_0 = 2N_c \left(q_0 + q_1(a/L) + q_2(a/L)^2 + 2d_1 \log(L/a) \right)$$

c_{10}	1	c_{40}	$-0.0040(82)$
c_{11}	0	c_{41}	$0.08(17)$
c_{12}	0	c_{42}	$-0.33(73)$
c_{20}	$-0.19(14)$	q_0	$0.34(48)$
c_{21}	$1.3(3.0)$	q_1	$-0.4(3.6)$
c_{22}	$-6.8(12.8)$	q_2	$5.3(12.4)$
c_{30}	$0.033(56)$	d_1	$-0.067(60)$
c_{31}	$-0.4(1.1)$	a_0	$1200(2600)$
c_{32}	$1.8(4.8)$	m_0	$1.5(0.3)$
N_{dof}	70	χ^2/N_{dof}	$1.99(1.09)$

$N_f = 8$ Data with Fits



Renormalization Group (Step Scaling)

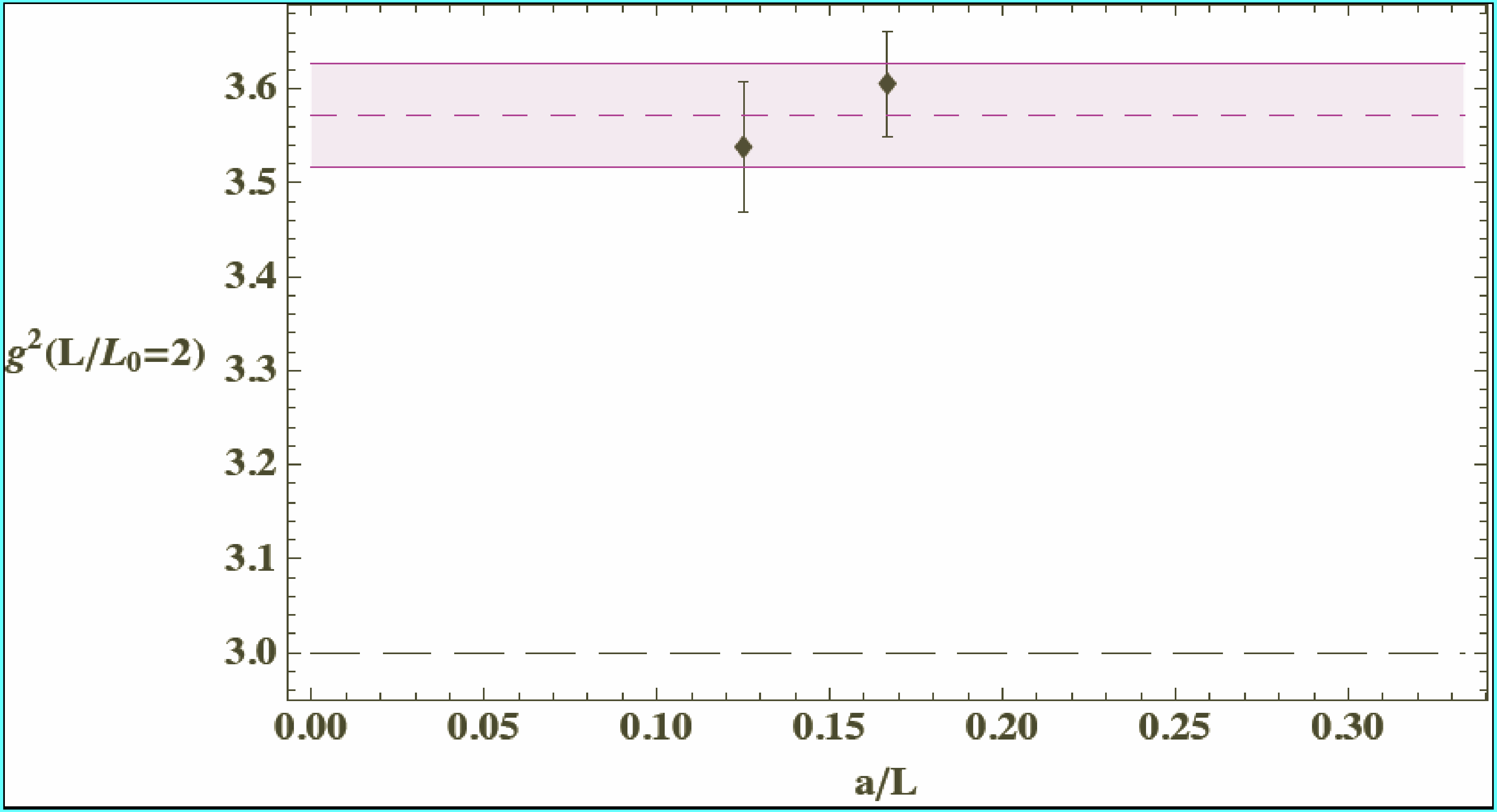
$$\bar{g}^2\left(g_0^2, \frac{a}{L}\right) = \bar{g}^2\left(\bar{g}^2(L_0), \frac{L}{L_0}, \frac{a}{L_0}\right)$$

$$g_0^2 \xrightarrow{a\to 0} 1/\ln(L_0/a)$$

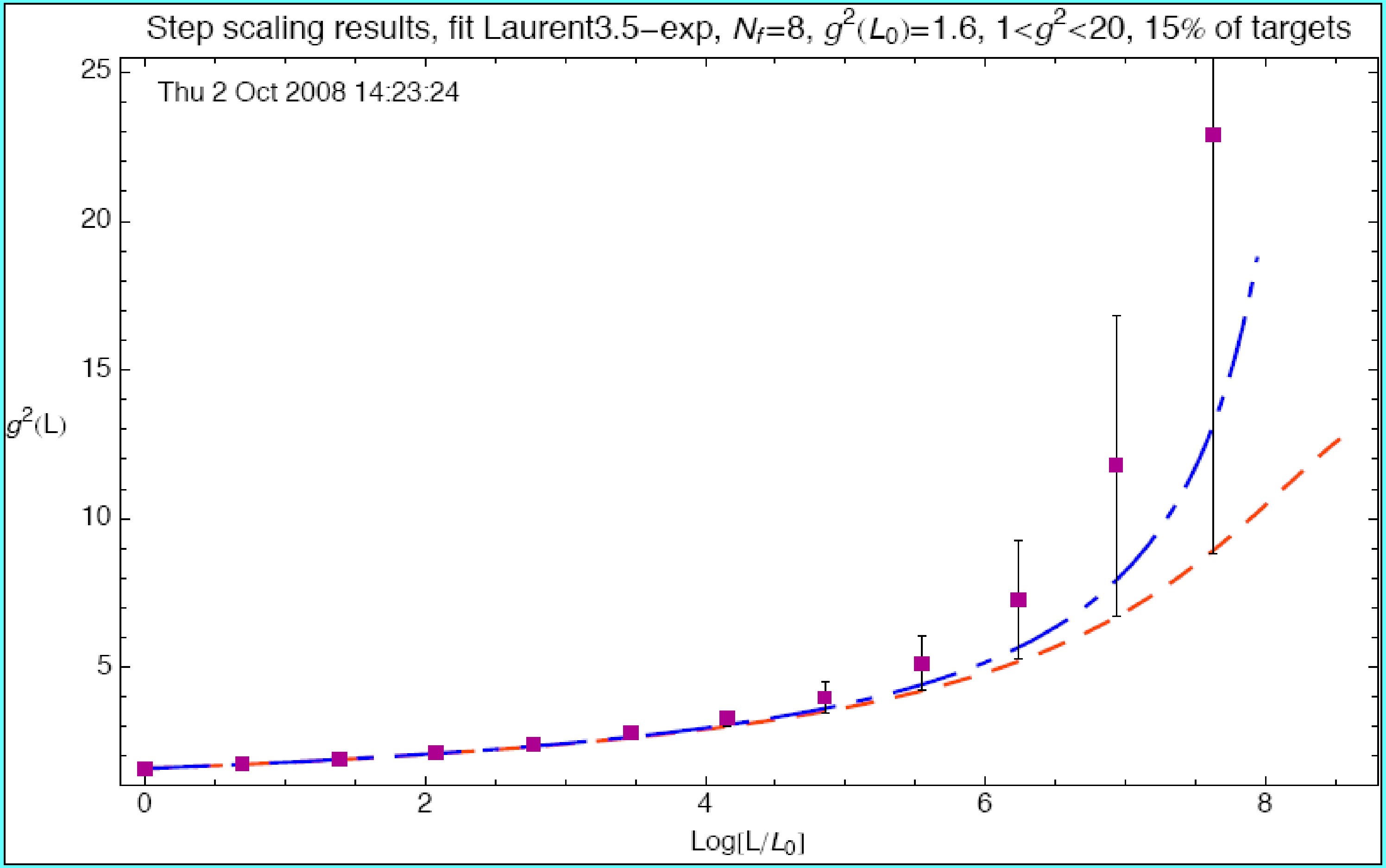
$$\xrightarrow{a/L_0\to 0} \bar{g}^2\left(\bar{g}^2(L_0), \frac{L}{L_0}\right) \equiv \bar{g}^2\left(\frac{L}{L_0}\right) \qquad \left(\frac{L}{L_0} = 2\right)$$

$$\bar{g}^2\left(\bar{g}^2(L), \frac{L'}{L}, \frac{a}{L}\right) \xrightarrow{\frac{a}{L}\to 0} \bar{g}^2\left(\frac{L'}{L}\right) \qquad \left(\frac{L'}{L} = 2\right)$$

N_f=8 Extrapolation Curve



N_f = 8 Continuum Running



N_f = 8 Features

1. No evidence for IRFP or even inflection point up through $\bar{g}^2(L) \approx 15$.

2. Exceeds rough estimate $\left(\alpha_c^* / \pi \approx 1/4\right)$ of strength required to break chiral symmetry, and therefore produce confinement. Must be confirmed by direct lattice simulations.

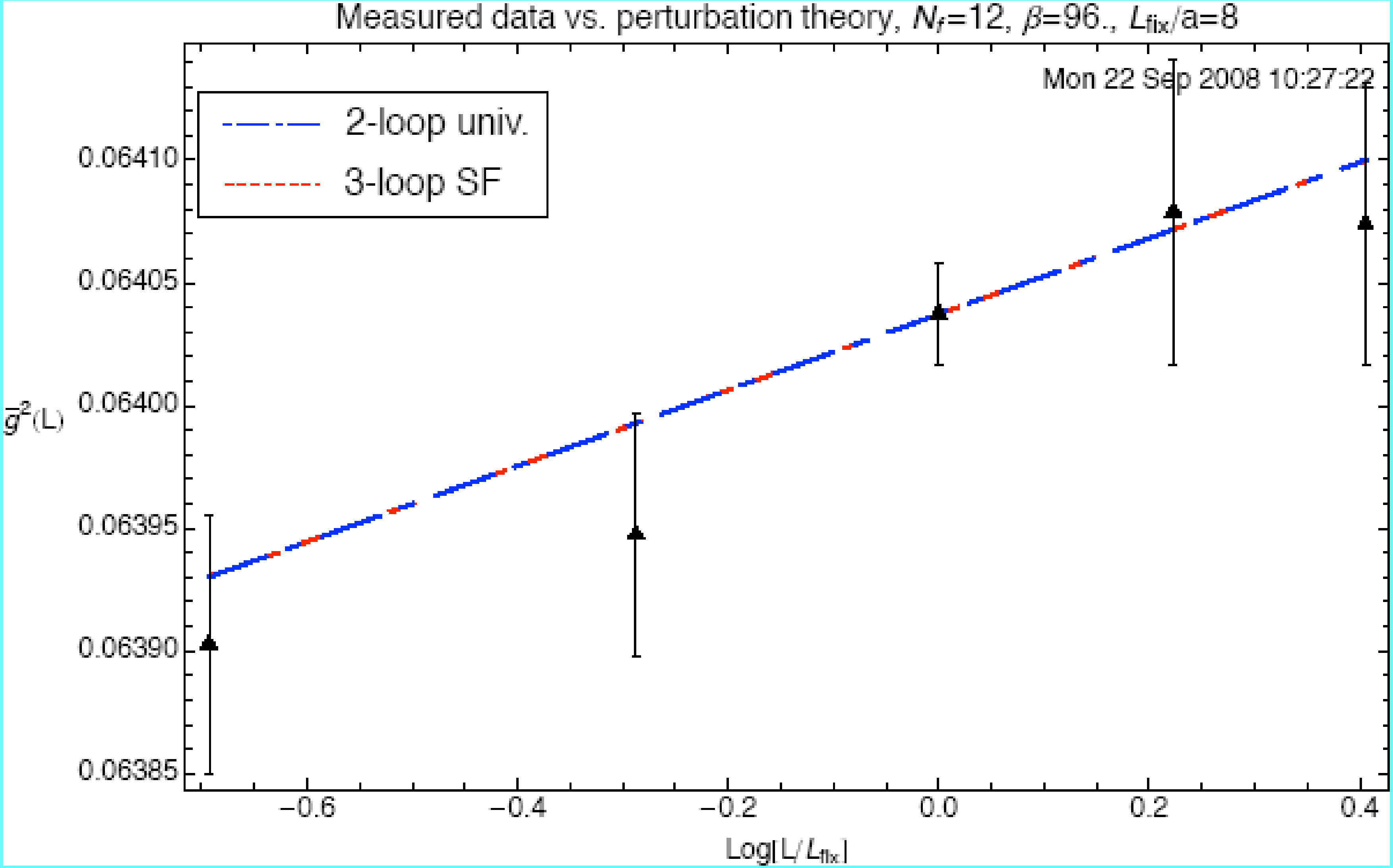
3. Rate of growth exceeds 3 loop perturbation theory.

4. Behavior similar to quenched theory [ALPHA N.P. Proc. Suppl. 106, 859 (2002)] and N_f=2 theory [ALPHA, N.P. B713, 378 (2005)], but slower growth as expected.

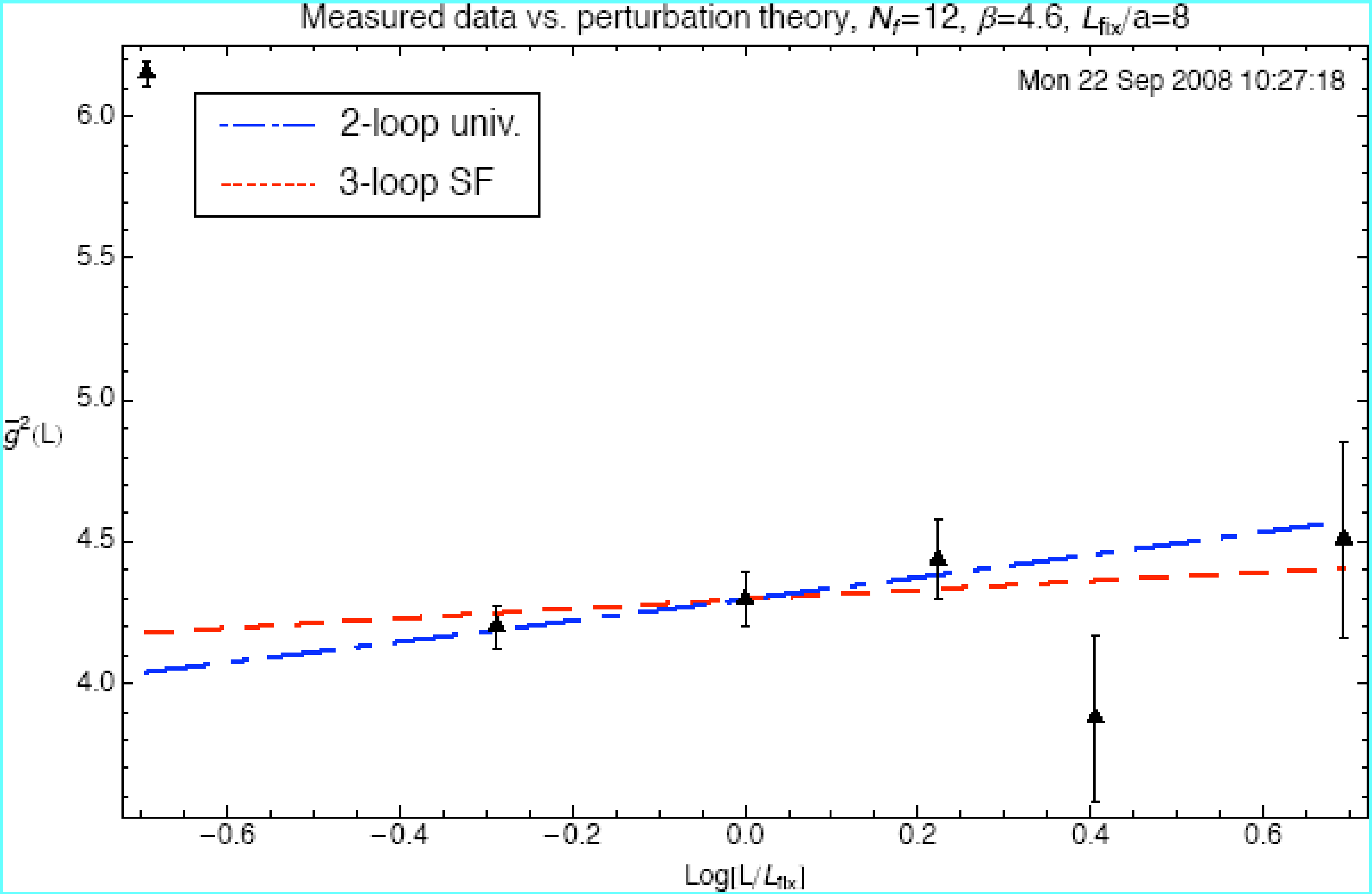
$N_f = 12$ Data

Blue (Red) numbers indicate the number of trajectories collected for • =• + 1 (• • 1).

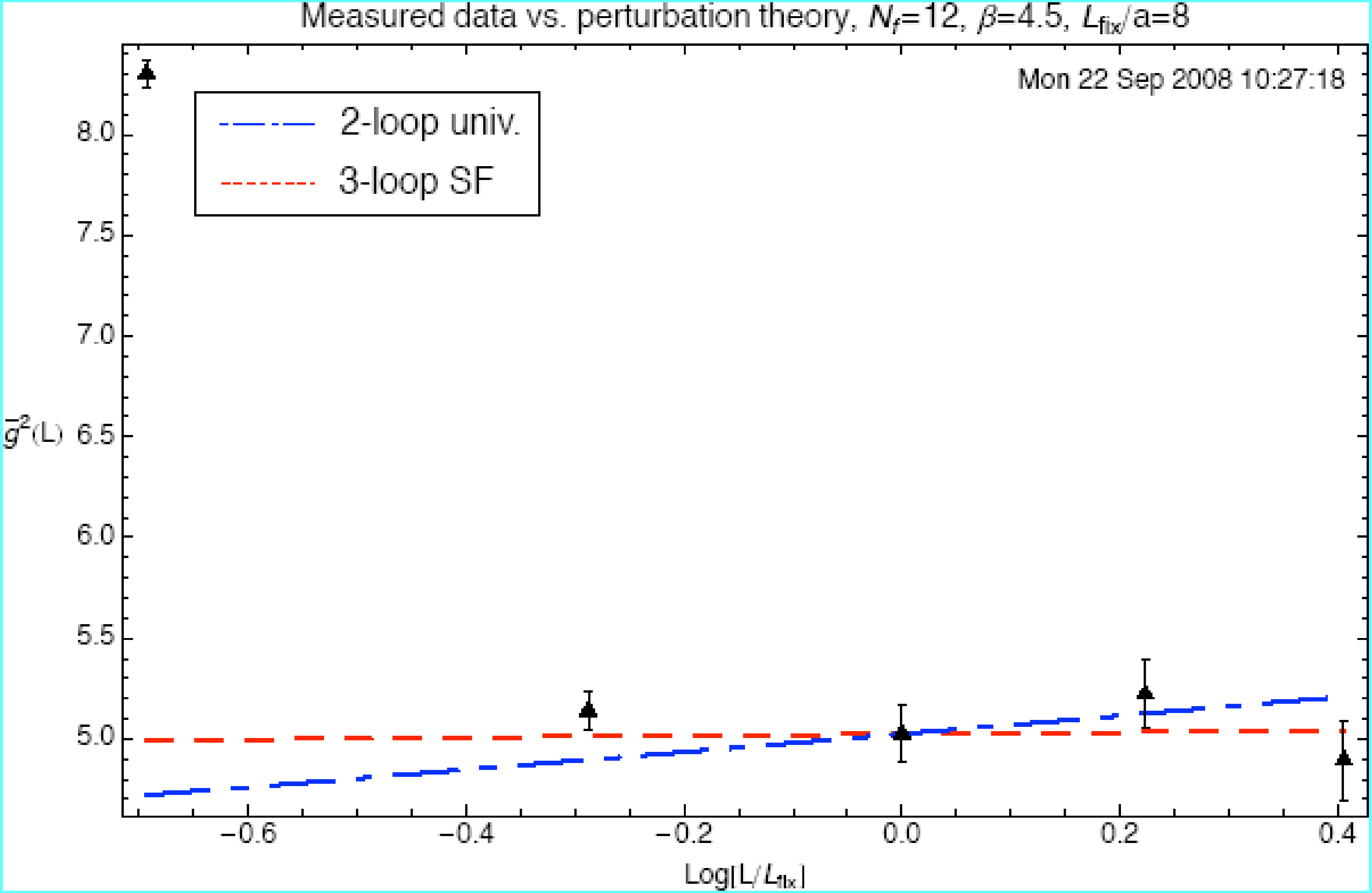
•²(•)					• • •								
•	4	• traj	6	• traj	8	• traj	10	• traj	12	• traj	16	• traj	20	• traj
4.10			66.0(9.7)	80• 81•	25.4(1.5)	76• 82•								
4.13			30.0(1.9)	80• 80•	22.8(2.6)	48• 62•								
4.15			26.6(0.9)	81• 81•	15.80(72)	81• 81•	15.61(98)	59• 79•						
4.18			17.94(85)	82• 82•	12.28(46)	81• 81•	12.75(83)	60• 82•						
4.20	-079(8800)	94• 160•	14.57(86)	82• 82•	11.3(0.4)	82• 82•	13.5(2.0)	50• 54•						
4.25	70.8(7.0)	162• 136•	10.53(32)	71• 75•	9.65(69)	49• 60•	9.58(32)	82• 82•						
4.30	29.91(91)	162• 178•	8.48(22)	82• 82•	7.28(27)	58• 58•	7.93(49)	53• 44•	6.65(53)	51• 52•	7.95(57)	42• 51•		
4.35	18.98(38)	162• 162•	6.94(15)	82• 82•	6.79(22)	82• 82•	6.76(22)	72• 82•	6.31(72)	70• 76•				
4.40	13.70(28)	162• 162•	6.023(73)	82• 82•	5.98(13)	82• 82•	6.13(35)	25• 36•	5.95(59)	79• 82•				
4.45	10.44(17)	162• 162•	5.3(0.1)	82• 82•	5.32(13)	82• 82•	4.99(16)	68• 82•						
4.50	8.300(72)	162• 162•	5.139(96)	96• 96•	5.02(14)	84• 98•	5.22(17)	85• 84•	4.9(0.2)	81• 83•				
4.60	6.152(45)	162• 162•	4.200(76)	82• 82•	4.297(98)	82• 82•	4.44(14)	81• 81•	3.93(29)	78• 74•	4.51(34)	40• 40•		
4.70	4.987(43)	162• 162•	3.842(54)	82• 80•	3.731(51)	86• 87•	3.657(98)	22• 22•	3.82(26)	82• 82•	5.44(42)	60• 61•		
4.80	4.244(23)	162• 162•	3.469(34)	82• 81•	3.507(56)	82• 82•	3.61(13)	39• 38•						
4.90	3.705(21)	162• 162•	3.105(29)	116• 132•	3.191(48)	82• 82•	3.207(57)	74• 79•						
5.00	3.346(23)	121• 40•	2.896(27)	54• 40•	2.918(35)	53• 89•	3.002(72)	40• 40•	3.142(93)	43• 43•	3.11(16)	65• 37•		
5.10	3.061(28)	40• 40•	2.731(34)	36• 52•	2.856(59)	41• 42•	2.674(14)	41• 41•						
5.20	2.825(25)	41• 40•	2.546(12)	36• 56•	2.568(25)	53• 89•								
5.30	2.618(20)	40• 40•	2.464(27)	36• 56•	2.471(38)	42• 42•	2.439(12)	41• 41•						
5.40	2.481(18)	41• 40•	2.325(20)	36• 56•	2.3104(95)	42• 42•								
5.50	2.343(15)	40• 40•	2.1965(95)	36• 56•	2.279(36)	32• 40•	2.238(17)	42• 42•	2.247(23)	54• 41•	2.307(19)	41• 41•	2.21(12)	39• 39•
5.60	2.238(14)	41• 40•	2.0978(41)	36• 56•	2.1161(74)	42• 42•								



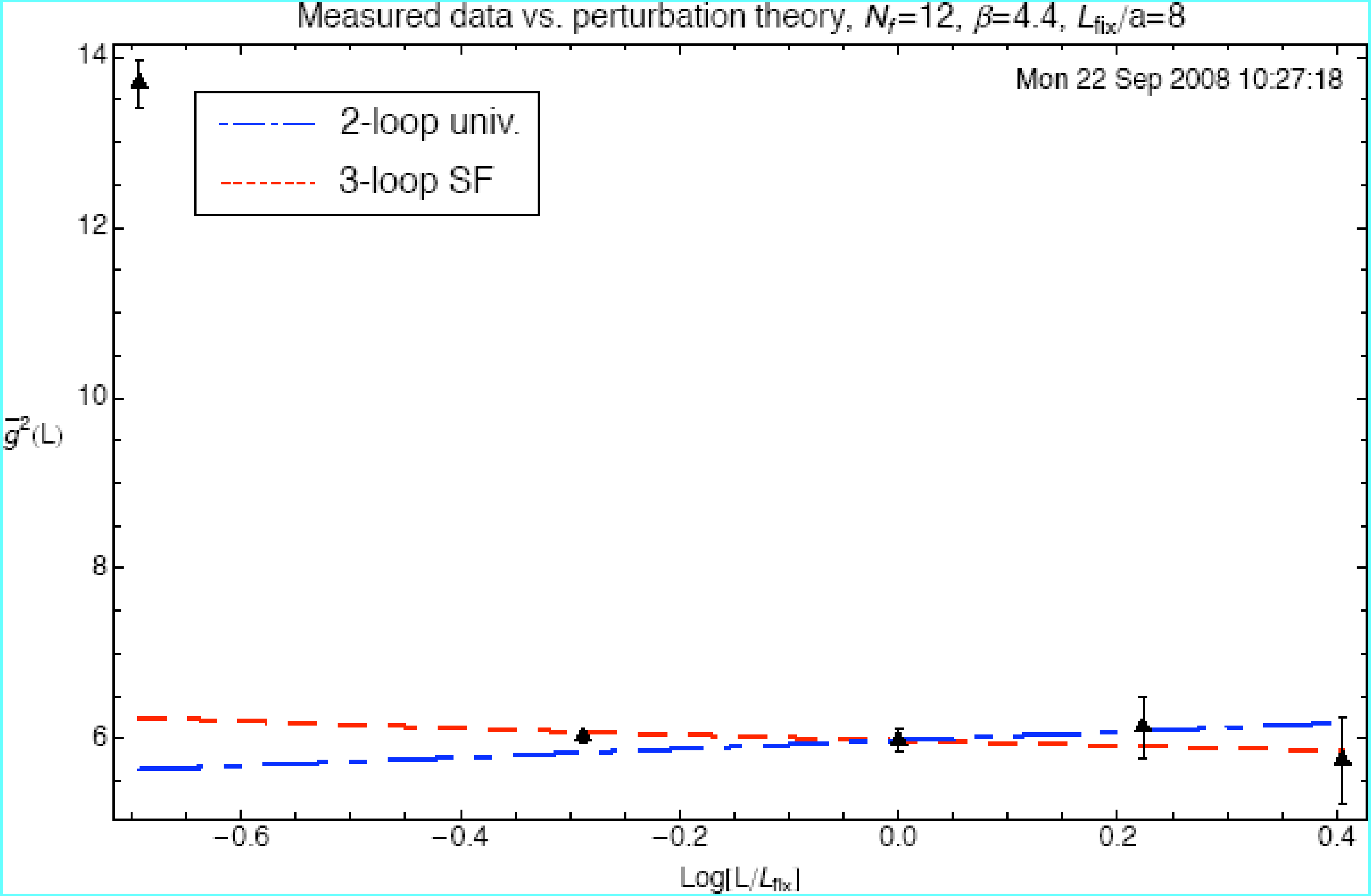
Measured data vs. perturbation theory, N_f=12, β=96., L_{fix}/a=8
Mon 22 Sep 2008 10:27:22
2-loop univ.
3-loop SF
$\bar{g}^2$(L)
0.06410
0.06405
0.06400
0.06395
0.06390
0.06385
−0.6
−0.4
−0.2
0.0
0.2
0.4
Log[L/L_{fix}]



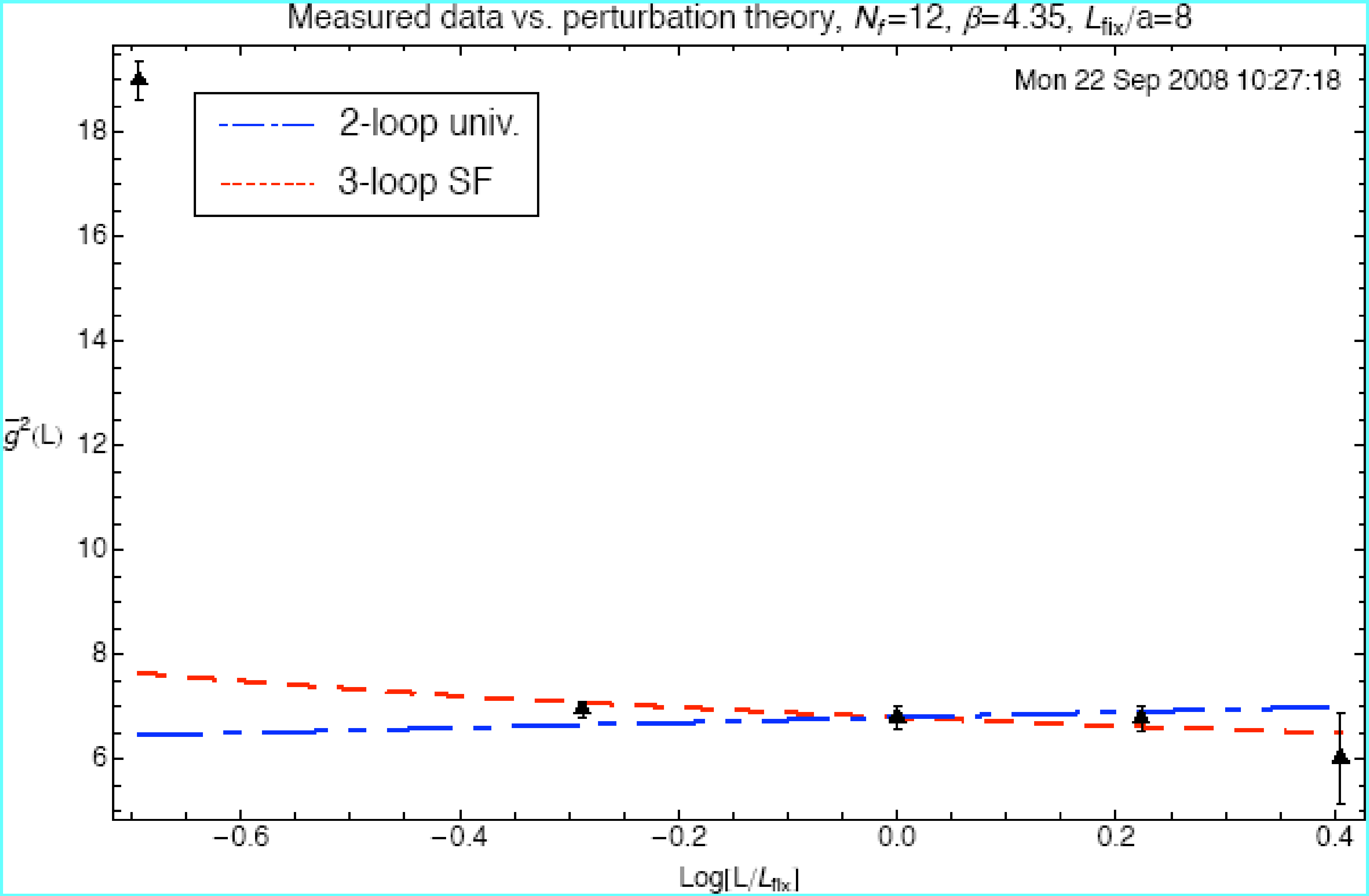
Measured data vs. perturbation theory, N_f=12, β=4.6, L_{fix}/a=8
Mon 22 Sep 2008 10:27:18
2-loop univ.
3-loop SF
6.0
5.5
5.0
4.5
4.0
$\bar{g}^2$(L)
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
Log[L/L_{fix}]



Measured data vs. perturbation theory, N_f=12, β=4.5, L_{fix}/a=8
Mon 22 Sep 2008 10:27:18
2-loop univ.
3-loop SF
$\bar{g}^2$(L)
8.0
7.5
7.0
6.5
6.0
5.5
5.0
−0.6
−0.4
−0.2
0.0
0.2
0.4
Log[L/L_{fix}]

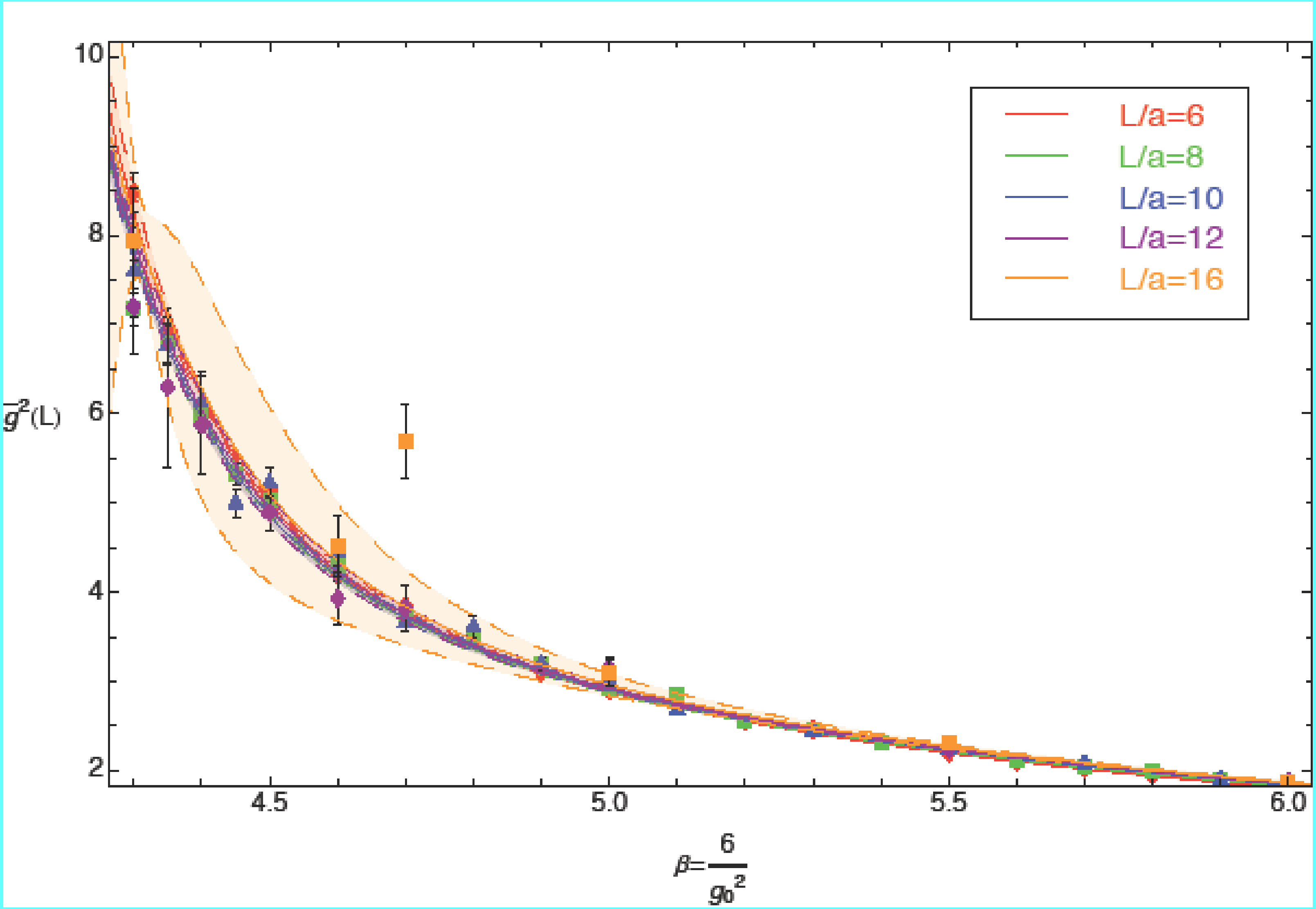


Measured data vs. perturbation theory, N_f=12, β=4.4, L_{fix}/a=8
Mon 22 Sep 2008 10:27:18
2-loop univ.
3-loop SF
$\bar{g}^2$(L)
14
12
10
8
6
−0.6
−0.4
−0.2
0.0
0.2
0.4
Log[L/L_{fix}]

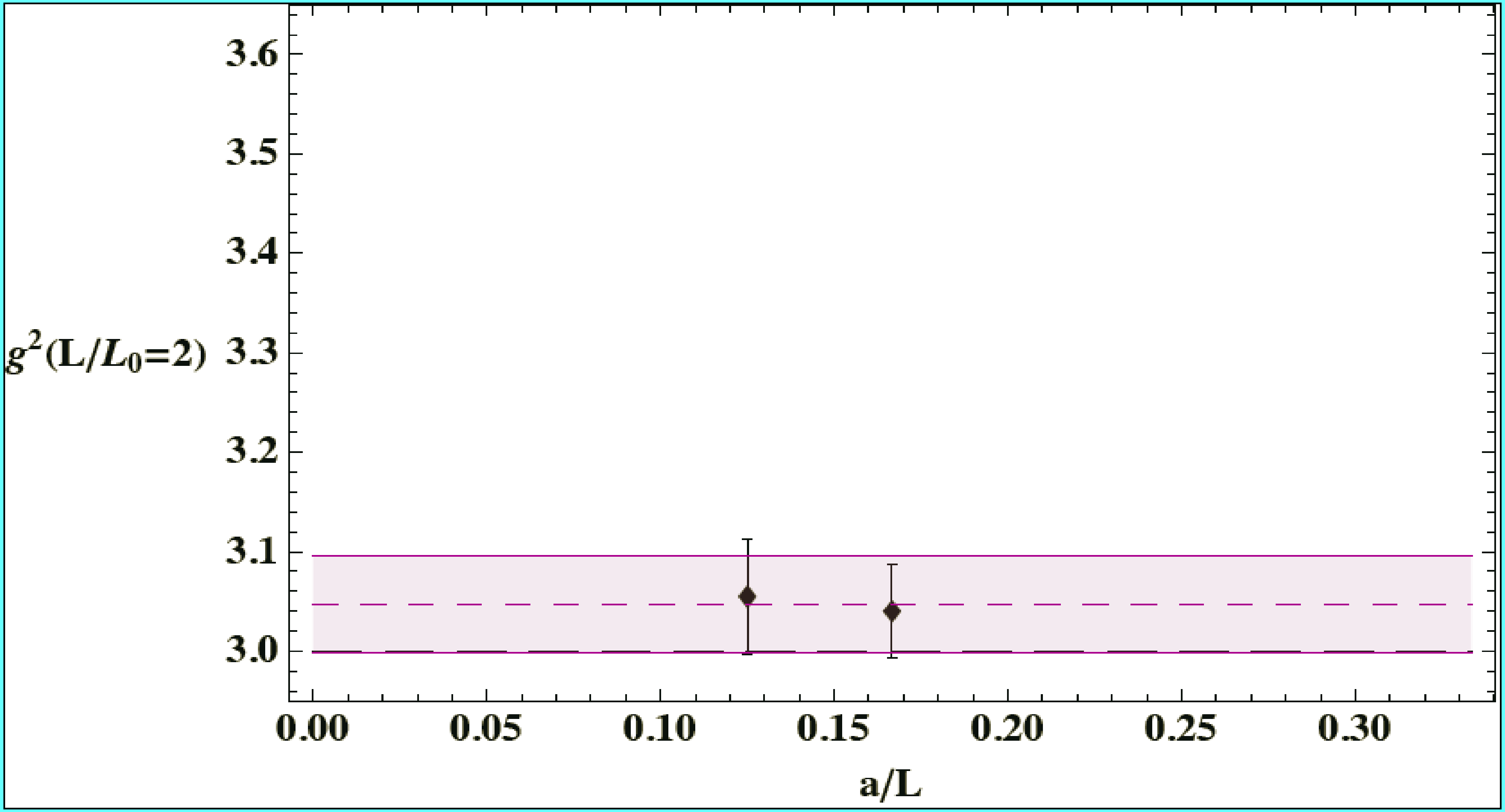


Measured data vs. perturbation theory, N_f=12, β=4.35, L_{fix}/a=8
Mon 22 Sep 2008 10:27:18
2-loop univ.
3-loop SF
$\bar{g}^2$(L)
18
16
14
12
10
8
6
−0.6
−0.4
−0.2
0.0
0.2
0.4
Log[L/L_{fix}]

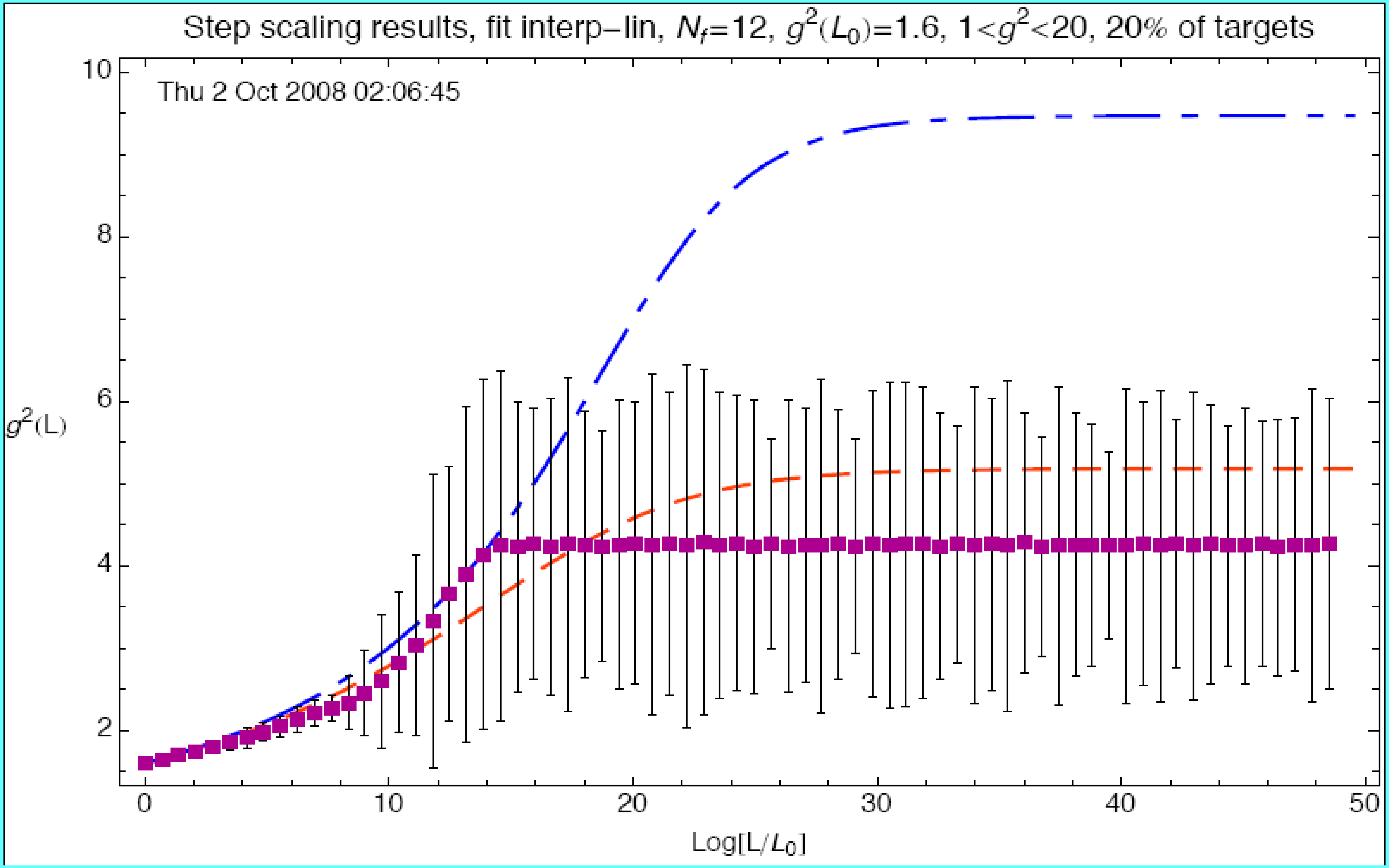
$N_f = 12$ Data with Fits



N_f=12 Extrapolation Curve



N_f = 12 Continuum Running



Conclusions

1. First lattice evidence that for an SU(3) gauge theory with N_f Dirac fermions in the fundamental representation

   $8 < N_{fc} < 12$

2. N_f=12: Relatively weak IRFP

3. N_f=8: Confinement and chiral symmetry breaking – in disagreement with Iwasaki et al

Employing the Schroedinger functional running coupling defined <u>at</u> the box boundary L

Things to Do

1. Refine the simulations at N_f = 8 and 12 and examine other values such as N_f =10.

2. Study the phase transition as a function of N_f.

3. Consider other gauge groups and representation assignments for the fermions

4. Examine physical quantities such as the static potential (Wilson loop) and correlation functions.

5. Examine chiral symmetry breaking directly:

<ψ ψ> at zero temperature

6. Apply to BSM Physics. Is S naturally small as $N_f \to N_{fc}$ due to approximate parity doubling?

$$S\left(m_{H,ref}\right) = 4\int_0^{\infty} \frac{ds}{s}\left\{\left[\mathrm{Im}\,\Pi_{VV}(s) - \mathrm{Im}\,\Pi_{AA}(s)\right] - \frac{1}{48\pi}\left[1 - \left(1 - \frac{m_{H,ref}}{s}\right)^3 \theta\left(s - m_{H,ref}^2\right)\right]\right\}$$

Includes the contribution of the [N_f^2 - 1 - 3] pseudo-Nambu-Goldstone bosons present in the model.

Further studies of the conformal window

- Study of asqtad $N_f = 8, 12$ finite temperature transition.[16] Indications of a physical confined phase for $N_f = 8$ and a bulk transition for $N_f = 12$.
- Columbia zero temperature spectrum study of staggered $N_f = 8$ with DBW2 gauge action.[17] Clear indications of confined spectrum properly scaling towards continuum limit.
- Study of staggered $N_f = 8, 12$ Dirac eigenvalue distributions underway for comparison with RMT.[18] Stout-link smearing important for degeneracies. Looking for the ϵ-regime: $m_\pi \ll L^{-1} \ll f_\pi$.
- A promising new scheme being developed to compute the running coupling from Wilson loops in a finite box.[19] Direct simulation at zero quark mass possible with twisted BC's.

[16] Deuzeman, Lombardo, Pallante, arXiv:0804.2905 [hep-lat]
[17] Xin and Mawhinney, in preparation.
[18] K. Holland *et al.*, in preparation.
[19] E. Bilgici *et al.*, arXiv:0808.2875 [hep-lat]

Conformal windows for higher color representations

- $N_f = 2$ Wilson fermions in the **6** of SU(3) may have an IRFP[20] and a novel deconfined yet chirally-broken phase. Explorations continue.
- Eigenvalue distributions of $N_f = 2$ overlap fermions in the **6** of SU(3) were studied on small volumes at fixed zero topology. Do not fit RMT predictions for χSB. f_π not measured yet, so may not be in ϵ-regime.
- Three groups are studying SU(2) with $N_f = 2$ adjoint Wilson fermions [21] [22] [23] and producing consistent results. Clear evidence for bulk transition at $\beta = 2$. For $\beta > 2$ vector mesons seem very light at small quark masses. Is it a finite volume effect?

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[20] Shamir, Svetitsky and DeGrand, arXiv:0803.1707 [hep-lat]
[21] Del Debbio, Patella and Pica, arXiv:0805.2058 [hep-lat]
[22] Catterall, Giedt, Sannino and Schneible, arXiv:0807.0792 [hep-lat]
[23] Hietanen, Rantaharju, Rummukainen, and Tuominen, in preparation

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Lattice Strong Dynamics

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