The effective action in Einstein-Maxwell theory

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Abstract. Considerable work has been done on the one-loop effective action in combined electromagnetic and gravitational fields, particularly as a tool for determining the properties of light propagation in curved space. After a short review of previous work, I will present some recent results obtained using the worldline formalism. In particular, I will discuss various ways of generalizing the QED Euler-Heisenberg Lagrangians to the Einstein-Maxwell case.

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THE WORLDLINE FORMALISM IN QED

Let us start with the "worldline" representation of the one-loop effective action in spinor QED [1, 2]

$$\Gamma(A) \quad = \quad -\frac{1}{2}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}\int_{x(T)=x(0)}\mathscr{D}x(\tau)\int_{\psi(T)=-\psi(0)}\mathscr{D}\psi(\tau)\,e^{-\int_0^T d\tau L[x(\tau)]} \tag{1}$$

Here m and T are the mass and proper time of the loop fermion, and the "worldline Lagrangian" L is given by

$$L \quad = \quad \frac{1}{4}\dot{x}^2 + ie\dot{x}^\mu A_\mu + \frac{1}{2}\psi\cdot\dot{\psi} - ie\psi^\mu F_{\mu\nu}\psi^\nu \tag{2}$$

The $x(\tau)$ part of the double path integral in (1) runs over all closed trajectories in spacetime with fixed periodicity in T, and by itself gives the effective action for a scalar loop (up to the normalization). The $\psi(\tau)$ integral represents the spin degree of freedom, and is over antiperiodic Grassmann functions, obeying $\psi(\tau_1)\psi(\tau_2) = -\psi(\tau_2)\psi(\tau_1)$ and $\psi(T) = -\psi(0)$. Similar worldline representations can be written for the effective action with open scalar/spinor lines, at the multiloop level, and for other field theories; see [3] for a review. However, it is only during the last fifteen years that such representations have gained some popularity for actual state-of-the-art calculations. By now a number of different techniques have been developed for the evaluation of worldline path integrals. We will follow the "string-inspired" approach [4, 5], where one manipulates the path integral into gaussian form, and then performs those gaussian integrals using worldline correlators adapted to the periodicity conditions,

$$
\begin{aligned}
\langle x^{\mu}(\tau_1)x^{\nu}(\tau_2)\rangle &= -G_B(\tau_1,\tau_2)\,\delta^{\mu\nu}, \quad G_B(\tau_1,\tau_2) = |\tau_1-\tau_2| - \frac{\left(\tau_1-\tau_2\right)^2}{T} - \frac{T}{6} \\
\langle \psi^{\mu}(\tau_1)\psi^{\nu}(\tau_2)\rangle &= \frac{1}{2}G_F(\tau_1,\tau_2)\,\delta^{\mu\nu}, \quad G_F(\tau_1,\tau_2) = \mathrm{sign}(\tau_1-\tau_2)
\end{aligned}
\tag{3}
$$

This procedure leads, for example, with little effort to the following "Bern-Kosower master formula" [6] for the one-loop N - photon amplitude in scalar QED:

$$
\begin{aligned}
\Gamma[\{k_i,\varepsilon_i\}] &= (-ie)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \\
&\times \exp\Big\{ \sum_{i,j=1}^N \big[\frac{1}{2} G_{Bij} k_i\cdot k_j + i\dot{G}_{Bij} k_i\cdot\varepsilon_j + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j\big] \Big\} \,|_{\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)}
\end{aligned}
\tag{4}
$$

Here k_i and ε_i are the momentum and polarization of the ith photon, and each τ_i integral represents one photon leg moving around the loop. The notation $\mathrm{lin}(\varepsilon_1,\dots,\varepsilon_N)$ means that only terms linear in all polarization vectors are to be kept after expanding the exponential. Apart from the worldline Green's function $G_B(\tau_i,\tau_j)$, which we abbreviate by G_{Bij}, also its first and second derivatives appear, $\dot{G}_{B12} = \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}$, $\ddot{G}_{B12} = 2\delta(\tau_1-\tau_2) - \frac{2}{T}$. The factor $(4\pi T)^{-\frac{D}{2}}$ in (4) represents the free path integral determinant in D dimensions.

The corresponding representation of the N photon amplitude for the spinor loop case differs from (4) (apart from a factor of -2) only by additional terms from the spin path integral in (1). Those terms can be inferred from the scalar loop integrand through a certain pattern matching rule [3, 5, 6].

A major advantage of the worldline formulation of QED is that it allows one to include an external constant field $F_{\mu\nu}$ in a particularly efficient way [7, 8]. Effectively, the integral representation of a scalar or spinor QED amplitude in such a constant external field is obtained from the corresponding one in vacuum by the following replacements of the worldline Green's functions and determinants,

$$
\begin{aligned}
G_B(\tau_1,\tau_2) &\rightarrow \mathcal{G}_B(\tau_1,\tau_2) = \frac{1}{2(eF)^2}\left(\frac{eF}{\sin(eFT)}\mathrm{e}^{-ieFT\dot{G}_{B12}} + ieF\dot{G}_{B12} - \frac{1}{T}\right) \\
G_F(\tau_1,\tau_2) &\rightarrow \mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\,\frac{\mathrm{e}^{-ieFT\dot{G}_{B12}}}{\cos(eFT)}
\end{aligned}
\tag{5}
$$

(the trigonometric expressions are to be understood as power series in the field strength matrix),

$$
\begin{aligned}
(4\pi T)^{-\frac{D}{2}} &\rightarrow (4\pi T)^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\sin eFT}{eFT}\right] && \text{(Scalar QED)}\\
(4\pi T)^{-\frac{D}{2}} &\rightarrow (4\pi T)^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\tan eFT}{eFT}\right] && \text{(Spinor QED)}
\end{aligned}
\tag{6}
$$

In particular, applying these changes in (4) yields a corresponding master formula for the N - photon amplitudes in a constant field [7, 8]. This master formula, and its extension to spinor QED, have been used for comparatively easy recalculations of the scalar and spinor QED vacuum polarization tensors [9], as well as of the photon splitting amplitudes in a magnetic field [10]. The determinant factors (6) by themselves (i.e., the $N = 0$ case) yield, after renormalization, the well-known effective Lagrangians of Weisskopf and Schwinger [11] and Euler-Heisenberg [12],

$$
\begin{aligned}
\mathscr{L}_{\rm scal}(F) &= \frac{1}{16\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\sinh(eaT)\sin(ebT)} + \frac{e^2}{6}(a^2-b^2)T^2 - 1\right]\\
\mathscr{L}_{\rm spin}(F) &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]
\end{aligned}
\tag{7}
$$

Here a,b are the two invariants of the Maxwell field, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2 - b^2 = B^2 - E^2, \quad ab = \mathbf{E}\cdot\mathbf{B}$.

A further extension to the two-loop level has been extensively applied to the study of the two-loop corrections to the effective Lagrangians (7) [8, 13, 14].

See also [15, 16] for the calculation of derivative corrections to the effective Lagrangian at the one-loop level. Here the gaussian form of the path integral is reached by Taylor expanding the background field at the loop center of mass, usually in Fock-Schwinger gauge to achieve manifest covariance.

GENERALIZATION TO GRAVITATIONAL BACKGROUNDS

To include an additional background gravitational field, naively one might replace

$$
S_0 = \frac{1}{4}\int_0^T d\tau \dot{x}^2 \quad\rightarrow\quad \frac{1}{4}\int_0^T d\tau \dot{x}^\mu g_{\mu\nu}(x(\tau))\dot{x}^\nu \tag{8}
$$

The usual expansion around flat space $g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}$ would then yield a graviton vertex operator $\varepsilon_{\mu\nu}\int_0^T d\tau \dot{x}^\mu \dot{x}^\nu\, \mathrm{e}^{ik\cdot x}$. However, using this operator in a formal gaussian integration leads to worldline integrands contaning ill-defined expressions such as $\delta(0), \delta^2(\tau_i - \tau_j), \ldots$. This comes not unexpected, since path integration in curved space is a subject notorious for its mathematical subtleties even in nonrelativistic quantum mechanics (see., e.g., [17] and refs. therein). Fortunately, during the past decade these issues have been intensively studied, and a consistent formalism has emerged for the calculation of worldline path integrals in general electromagnetic-gravitational backgrounds [18]. A detailed account of this recent development has been given in [19]. Here we can only mention that the main difficulty arises from the nontriviality of the path integral measure in curved space, which leads to spurious UV divergences. Those can be removed by regularization, but leave an ambiguity which has to be removed by counterterms to the worldline Lagrangian. Those are regularization-dependent, and in general non-covariant, the only known exception being one-dimensional dimensional regularization. A further problem consists in the zero mode which appears in the perturbative expansion of the path integral. In the string-inspired approach this zero mode must be fixed as the loop center-of-mass, but this leads to a nontrivial Fadeev-Popov type determinant in the path integral.

Concerning previous applications of the worldline formalism in curved space, let us mention (i) the calculation of various types of anomalies (see [19] and refs therein) (ii) the (re)calculation of the one loop graviton self energy due to a scalar loop [20], spinor loop [21], and loops due to vector and arbitrary differential forms [22] (iii) the first calculation of the one loop photon-graviton amplitudes in a constant electromagnetic field [23] (iv) the one loop photon vacuum polarization in a generic gravitational background due to a scalar loop in the semiclassical approximation [24].

THE EFFECTIVE ACTION FOR EINSTEIN-MAXWELL THEORY

Pure Einstein-Maxwell theory is described by the action

$$\Gamma^{(0)}[g,A] = \int d^4x \, \sqrt{g} \left(\frac{1}{2\kappa^2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \tag{9}$$

(here and in the following we absorb the coupling e into F). In 1980, Drummond and Hathrell [25] studied the one-loop corrections $\Gamma^{(1)}_{\rm spin}[g,A]$ to this action due to a spinor loop, and calculated the terms in it quadratic in the electromagnetic field, and linear in the curvature:

$$\mathscr{L}^{(DH)}_{\rm spin} = \frac{1}{180(4\pi)^2 m^2} \left(5RF^2_{\mu\nu} - 26R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + 2R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + 24(\nabla^\alpha F_{\alpha\mu})^2 \right) \tag{10}$$

The point of singling out these terms is that they contain the information on the modifications of light propagation by weak gravitational fields in the limit of zero photon energies. In the following, our goal is to generalize this result to include the effect of a constant external field nonperturbatively, i.e., we are looking for the gravitational corrections to the Euler-Heisenberg Lagrangians (7) to linear order in the curvature. Here it must be said that those flat space Lagrangians could be defined in either of two equivalent ways: (i) by the constancy of the background field $F_{\mu\nu}$ (ii) by the property of carrying the full information on the low energy limits of the corresponding N - photon amplitudes. The lowest order gravitational corrections could be defined either by generalizing (i) to covariant constancy, or by generalizing (ii) by requiring that the effective Lagrangians should carry the information on the low energy limits of the amplitudes with N photons and with one graviton. These generalizations are not any more equivalent, and we will adopt (ii) here rather than (i) (for the effective Lagrangian defined by covariant constancy Avramidi has obtained a representation in terms of integrals over the holonomy group [26]).

With our definition of the generalized Euler-Heisenberg Lagrangian, we have to get all terms involving arbitrary powers of $F_{\mu\nu}$ and one factor of $R_{\mu\nu\kappa\lambda}$ or $\nabla_\mu\nabla_\nu$. As in the flat space case, the path integrals are gaussianized by a Taylor expansion at the loop center-of-mass x_0, made covariant by combining Fock-Schwinger gauge and Riemann normal coordinates [27]

$$
\begin{aligned}
A_\mu(x_0+y) &= -\frac{1}{2}F_{\mu\nu}(x_0)\,y^\nu - \frac{1}{3}F_{\mu\nu;\alpha}(x_0)\,y^\nu\,y^\alpha \\
&\quad -\frac{1}{8}\left[F_{\mu\nu;\alpha\beta}(x_0) + \frac{1}{3}R_{\alpha\mu}{}^{\lambda}{}_{\beta}(x_0)F_{\lambda\nu}(x_0)\right]y^\alpha\,y^\beta\,y^\nu + \cdots \\
g_{\mu\nu}(x_0+y) &= g_{\mu\nu}(x_0) + \frac{1}{3}R_{\mu\alpha\beta\nu}(x_0)y^\alpha y^\beta + \ldots.
\end{aligned}
\tag{11}
$$

Concentrating on the spinor loop case, the effective Lagrangian then is obtained in the following form,

$$
\mathscr{L}_{\rm spin} = -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\det{}^{-\frac{1}{2}}\left[\frac{\tan(FT)}{FT}\right]\Big\langle\,\mathrm{e}^{-S_{\rm int}}\Big\rangle_{S_0}
\tag{12}
$$

Here S_0 denotes the quadratic part of the worldline action, which is (after a rescaling to the unit circle)

$$
S_0 = \int_0^1 d\tau\left(\frac{1}{4T}g_{\mu\nu}(x_0)\dot{y}^\mu\dot{y}^\nu - \frac{i}{2}F_{\mu\nu}(x_0)\dot{y}^\mu y^\nu + \frac{1}{2}g_{\mu\nu}(x_0)\,\psi^\mu\,\dot{\psi}^\nu - iTF_{\mu\nu}(x_0)\,\psi^\mu\,\psi^\nu\right)
\tag{13}
$$

It yields again the generalized worldline Green's functions of (5), only that in taking powers of the field strength matrix the lowering and raising of indices involves the metric $g_{\mu\nu}(x_0)$. The interaction part involves the terms coming from the replacement (8), as well as a ghost part $S_{\rm gh}$ from the path integral measure, and a term S_{FP} representing the contribution from the Fadeev-Popov determinant mentioned above:

$$S_{\rm int} \;=\; S_{\rm grav}+S_{\rm gh}+S_{\rm em}+S_{\rm em,grav}+S_{FP} \tag{14}$$

$$\begin{aligned}
S_{grav}+S_{gh} &= \int_0^1 d\tau \Biggl\{ \frac{1}{12T} R_{\mu\alpha\beta\nu} y^\alpha y^\beta \Bigl[\dot y^\mu \dot y^\nu + a^\mu a^\nu + b^\mu c^\nu + 2\alpha^\mu \alpha^\nu\Bigr] \\
&\quad + \frac{1}{6} R_{\mu\alpha\beta\nu}\, y^\alpha y^\beta\, \psi^\mu\, \dot\psi^\nu + \frac{1}{6}(R_{\mu\alpha\lambda\beta}+R_{\mu\beta\lambda\alpha})\dot y^\alpha y^\lambda\, \psi^\mu\, \psi^\beta \Biggr\} \\
S_{em} &= \int_0^1 d\tau \Bigl[-\frac{i}{3} F_{\mu\nu;\alpha}\Bigl(\dot y^\mu y^\nu + 3T\psi^\mu\,\psi^\nu\Bigr) y^\alpha - \frac{i}{8} F_{\mu\nu;\alpha\beta}\Bigl(\dot y^\mu y^\nu + 4T\psi^\mu \psi^\nu\Bigr) y^\alpha y^\beta \Bigr] \\
S_{em,grav} &= -\frac{i}{24}\int_0^1 d\tau R_{\alpha\mu}{}^{\lambda}{}_{\beta} F_{\lambda\nu}\Bigl[\dot y^\mu y^\nu + 8T\psi^\mu \psi^\nu\Bigr] y^\alpha y^\beta \\
S_{FP} &= -\frac{1}{3}\int_0^1 d\tau\; \bar\eta_\mu R^\mu{}_{\alpha\beta\nu}\, y^\alpha y^\beta\, \eta^\nu
\end{aligned} \tag{15}$$

It is then a matter of simple combinatorics to arrive at our final result, an integral representation of the leading gravitational correction to the (unrenormalized) Euler-Heisenberg Lagrangian [28]:

$$\begin{aligned}
\mathscr{L}_{\rm spin} &= -\frac{1}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\, {\rm e}^{-m^2T} {\rm det}^{-1/2}\Bigl[\frac{\tan(FT)}{FT}\Bigr] \\
&\quad \times\Biggl\{ 1 + \frac{iT^2}{8} F_{\mu\nu;\alpha\beta}\, \mathscr{G}_{B11}^{\alpha\beta}\Bigl(\ddot{\mathscr{G}}_{B11}^{\mu\nu} - 2\mathscr{G}_{F11}^{\mu\nu}\Bigr) \\
&\quad + \frac{iT^2}{8}\left(F_{\mu\nu;\beta\alpha}+F_{\mu\nu;\alpha\beta}\right)\dot{\mathscr{G}}_{B11}^{\mu\beta}\dot{\mathscr{G}}_{B11}^{\nu\alpha} + \frac{T}{3} R_{\alpha\beta}\,\mathscr{G}_{B11}^{\alpha\beta} \\
&\quad - \frac{iT^2}{24} F_{\lambda\nu} R^\lambda{}_{\alpha\beta\mu}\,\Bigl(\ddot{\mathscr{G}}_{B11}^{\nu\mu}\,\mathscr{G}_{B11}^{\alpha\beta} + \dot{\mathscr{G}}_{B11}^{\alpha\mu}\,\dot{\mathscr{G}}_{B11}^{\nu\beta} + \dot{\mathscr{G}}_{B11}^{\beta\mu}\,\dot{\mathscr{G}}_{B11}^{\nu\alpha} + 4\,\mathscr{G}_{F11}^{\mu\nu}\,\mathscr{G}_{B11}^{\alpha\beta}\Bigr) \\
&\quad + \frac{T}{12} R_{\mu\alpha\beta\nu}\Bigl(\dot{\mathscr{G}}_{B11}^{\mu\alpha}\dot{\mathscr{G}}_{B11}^{\beta\nu} + \dot{\mathscr{G}}_{B11}^{\mu\beta}\dot{\mathscr{G}}_{B11}^{\alpha\nu} + \Bigl(\ddot{\mathscr{G}}_{B11}^{\mu\nu} - 2g^{\mu\nu}\delta(0)\Bigr)\mathscr{G}_{B11}^{\alpha\beta} \\
&\quad + \dot{\mathscr{G}}_{B11}^{\alpha\beta}\,\mathscr{G}_{F11}^{\mu\nu} + \dot{\mathscr{G}}_{B11}^{\nu\beta}\,\mathscr{G}_{F11}^{\mu\alpha} - \mathscr{G}_{B11}^{\alpha\beta}\,\Bigl(\dot{\mathscr{G}}_{F11}^{\mu\nu} - 2g^{\mu\nu}\delta(0)\Bigr)\Bigr) \\
&\quad - \frac{1}{6}T^3 F_{\alpha\beta;\gamma}F_{\mu\nu;\eta}\int_0^1 d\tau_1\Bigl(\dot{\mathscr{G}}_{B12}^{\alpha\nu}\,\dot{\mathscr{G}}_{B12}^{\beta\mu}\,\mathscr{G}_{B12}^{\gamma\eta} + \dot{\mathscr{G}}_{B12}^{\alpha\nu}\,\mathscr{G}_{B12}^{\beta\eta}\,\dot{\mathscr{G}}_{B12}^{\gamma\mu} \\
&\quad + \frac{3}{2}\,\mathscr{G}_{B12}^{\gamma\eta}\,\mathscr{G}_{F12}^{\alpha\mu}\,\mathscr{G}_{F12}^{\beta\nu}\Bigr)\Biggr\}
\end{aligned} \tag{16}$$

$(\tau_2 = 0)$.

As a check on (16), we have verified that an expansion to order RFF reproduces the result of Drummond-Hathrell up to total derivative terms:

$$\begin{aligned}
\mathscr{L}_{\rm spin} &= -\frac{1}{8\pi^2 m^2}\Big[-\frac{1}{72}RF_{\mu\nu}^2+\frac{1}{180}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} \\
&\qquad +\frac{1}{36}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}-\frac{1}{180}(\nabla_\alpha F_{\mu\nu})^2+\frac{1}{36}F_{\mu\nu}\Box F^{\mu\nu}\Big] \\
\mathscr{L}_{\rm spin}-\mathscr{L}_{\rm spin}^{(DH)} &= -\frac{1}{8\pi^2 m^2}\Big\{\frac{1}{36}\nabla^\alpha(F^{\mu\nu}F_{\mu\nu;\alpha})+\frac{1}{15}\Big[\nabla_\alpha(F_\mu{}^\alpha\nabla_\beta F^{\mu\beta})-\nabla_\beta(F_\mu{}^\alpha\nabla_\alpha F^{\mu\beta})\Big]\Big\}
\end{aligned} \tag{17}$$

As to possible applications of the Lagrangian (16), let us mention that it contains the information on (i) the one graviton - N photon amplitudes in the low energy limit (ii) the modified photon dispersion relations in the background of a strong electromagnetic and weak gravitational field (iii) the Schwinger pair production rate in such a field.

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