Steps toward Dyson-Schwinger equations for equal-time correlation functions

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Abstract. Our aim is to put the partially successful analytic noncovariant approaches to Coulomb gauge QCD on a firm and systematic basis. To this end, we develop a generating functional approach to the equal-time correlation functions. In fact, such a functional is given in terms of the vacuum wave functional, however, in a perturbative expansion of the equal-time correlation functions, the vacuum wave functional has to be known completely to the corresponding order. In general, we find many different contributions that correspond to one and the same Feynman diagram in the covariant theory. A remarkable simplification occurs on summing up these different contributions. We analyze the relatively simpler case of $\lambda\phi^4$ theory in detail and tentatively formulate new diagrammatic rules directly for the sum of all contributions that correspond to the same proper Feynman diagram.

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Dyson-Schwinger equations as a semi-analytical tool have given access, for the first time, to the deep infrared region of QCD (or Yang-Mills theory) in the Landau gauge [1]. In the so-called ghost dominance approximation, even a very simple analytical solution exists in the far infrared [2, 3]. Recent efforts have gone into repeating this success for QCD in the Coulomb gauge, the reason being that the color-Coulomb potential appearing in the Hamiltonian in the latter gauge in principle gives direct access to the long-range confining potential between color charges.

However, the usual covariant (four-dimensional) quantum field theoretic formulation becomes rather awkward in the Coulomb gauge. Partially successful analytic calculations so far have used alternative noncovariant approaches in three (spatial) dimensions [4], but a consistent solution of the corresponding nonperturbative equations including the Coulomb potential does not exist in the approximations presently employed. Nonetheless, an intriguing relation between Landau and Coulomb gauge QCD has been uncovered in the ghost dominance approximation [3, 5]: equal-time correlation functions in Coulomb gauge appear as the strict three-dimensional counterpart of the covariant correlation functions in (four-dimensional) Landau gauge. The latest numerical evaluation of equal-time correlation functions in the Coulomb gauge [6] seems to confirm this scenario.

Given this analogy between Landau and Coulomb gauge, it seems worth while to develop a representation of equal-time correlation functions that is analogous to covariant correlation functions, with the hope to find the analogue of the Dyson-Schwinger equations for the equal-time correlation functions. In such a systematic approach, one may expect to obtain consistent (approximate) solutions for Coulomb gauge QCD in the deep infrared region.

With this goal in mind, we will analyze in this contribution a functional integral representation of equal-time correlation functions in a slightly more general setting. For concreteness, we will write all the formulae for scalar $\lambda\phi^4$ theory which spares us many of the technical complications of QCD. All the structural results, however, are expected to carry over to Coulomb gauge QCD without change.

In fact, it is very simple to write down a functional integral for equal-time correlation functions, given that they are nothing but the the true vacuum expectation values of products of the field operators. The Schrödinger representation of the field theory directly yields for the n-point function in (3-)momentum space

$$\langle\phi(\mathbf{p}_1,t=0)\phi(\mathbf{p}_2,t=0)\cdots\phi(\mathbf{p}_n,t=0)\rangle = \int D[\phi]\,\phi(\mathbf{p}_1)\phi(\mathbf{p}_2)\cdots\phi(\mathbf{p}_n)\,|\psi(\phi)|^2\,, \quad (1)$$

where $\psi(\phi)$ is the true vacuum wave functional of the theory. The (absolute) square $|\psi(\phi)|^2$ then plays the rôle of the exponential of the negative Euclidean classical action in the corresponding representation of the covariant correlation functions (in Euclidean space).

In order to write down the functional integral for the equal-time correlation functions explicitly, we hence need an explicit expression for the vacuum wave functional. The analogy with the covariant theory suggests to make an exponential ansatz for this wave functional. We consider a full Volterra expansion of the exponent, but leave out all odd powers of ϕ in the ($\phi\to-\phi$) symmetric phase:

$$\psi(\phi) = \exp\Bigg(-\sum_{k=1}^{\infty}\frac{1}{(2k)!}\int\frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_{2k}}{(2\pi)^3}\,f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k}) \\ \times\phi(\mathbf{p}_1)\cdots\phi(\mathbf{p}_{2k})(2\pi)^3\delta(\mathbf{p}_1+\ldots+\mathbf{p}_{2k})\Bigg)\,. \quad (2)$$

When we insert this ansatz into the Schrödinger equation $H\psi(\phi)=E_0\psi(\phi)$ and equate the coefficient functions of corresponding powers of ϕ multiplying $\psi(\phi)$ on both sides of the Schrödinger equation, we obtain an infinite tower of equations for the coefficient functions in the Volterra expansion. The first few equations are

$$\big[f_2(\mathbf{p}_1)\big]^2 \equiv \big[f_2(\mathbf{p}_1,\mathbf{p}_2)\big]^2 \\ = m^2+\mathbf{p}_1^2+\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\,f_4(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{q},\mathbf{q}) \qquad (\text{with } \mathbf{p}_2=-\mathbf{p}_1)\,, \quad (3)$$

$$\big[f_2(\mathbf{p}_1)+\ldots+f_2(\mathbf{p}_4)\big]f_4(\mathbf{p}_1,\ldots,\mathbf{p}_4) \\ = \lambda+\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\,f_6(\mathbf{p}_1,\ldots,\mathbf{p}_4,-\mathbf{q},\mathbf{q})\,, \quad (4)$$

$$\big[f_2(\mathbf{p}_1)+\ldots+f_2(\mathbf{p}_6)\big]f_6(\mathbf{p}_1,\ldots,\mathbf{p}_6) \\ = -\big[f_4(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,-\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3)f_4(\mathbf{p}_4,\mathbf{p}_5,\mathbf{p}_6,\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3) \\ +(\text{9 permutations of the momenta})\big]+\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\,f_8(\mathbf{p}_1,\ldots,\mathbf{p}_6,-\mathbf{q},\mathbf{q})\,. \quad (5)$$

For a diagrammatic representation, we draw a “blob” with $2k$ external legs for $f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k})$ (taking the momenta as outgoing). The multiplication of two such coefficient functions with one momentum in common (with opposite signs), as in the first term on the right-hand side of Eq. (5), is represented as a contraction of the corresponding legs (without assigning a factor to the “propagator”). The last term in each of the equations is then represented as a loop, a contraction of two legs emerging from the same “blob”.

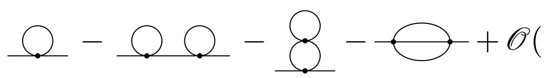
This tower of equations can be solved iteratively in a perturbative expansion. To this end, start with the vacuum functional of the noninteracting theory. With the notation $\omega_{\mathbf{p}} = (m^2+\mathbf{p}^2)^{1/2}$, this gives

$$f_2(\mathbf{p}) = \omega_{\mathbf{p}} + \mathcal{O}(\lambda) \equiv (—)^{-1} + \mathcal{O}(\lambda)\,, \tag{6}$$

introducing a diagrammatic representation for $\omega_{\mathbf{p}}$. Inserting this result in Eq. (4) and considering $f_6(\mathbf{p}_1,\ldots,\mathbf{p}_4,-\mathbf{q},\mathbf{q})$ to be of higher order, we obtain

$$f_4(\mathbf{p}_1,\ldots,\mathbf{p}_4) = \frac{\lambda}{\omega_{\mathbf{p}_1}+\ldots+\omega_{\mathbf{p}_4}} + \mathcal{O}(\lambda^2) \equiv -\times + \mathcal{O}(\lambda^2)\,. \tag{7}$$

We can use the latter result and feed it back into Eq. (3) in order to obtain the contribution to $f_2(\mathbf{p}_1)$ to order λ. According to our diagrammatic rules, we have to join two of the legs of the vertex corresponding to $f_4(\mathbf{p}_1,\mathbf{p}_2,-\mathbf{q},\mathbf{q})$ to form a loop in order to represent this contribution. We can also use our result (7) in Eq. (5) to find an expression for $f_6(\mathbf{p}_1,\ldots,\mathbf{p}_6)$ to order λ^2 [consistently with Eq. (7)], considering $f_8(\mathbf{p}_1,\ldots,\mathbf{p}_6,-\mathbf{q},\mathbf{q})$ there to be of higher order. Diagrammatically, this result for $f_6(\mathbf{p}_1,\ldots,\mathbf{p}_6)$ is represented as a contraction of two legs from two different f_4-vertices (we do not represent the division through $\omega_{\mathbf{p}_1}+\ldots+\omega_{\mathbf{p}_6}$ graphically). We then go on to substitute the result for $f_6(\mathbf{p}_1,\ldots,\mathbf{p}_6)$ back into Eq. (4) to obtain the one-loop corrections to $f_4(\mathbf{p}_1,\ldots,\mathbf{p}_4)$ which in turn, when used in Eq. (3), lead to

$$f_2(\mathbf{p}) = (—)^{-1} -$$

$$+\,\mathcal{O}(\lambda^3)\,. \tag{8}$$

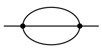
The signs in front of the diagrams are chosen merely for later convenience.

In general, assuming that the lowest-order contribution to the coefficient function $f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k})$ is of the order λ^{k-1}, we obtain a unique perturbative expansion of the coefficient functions by iteratively solving the tower of equations obtained from the Schrödinger equation for $\psi(\phi)$. We also get, from the rules stated above, a diagrammatic representation of all the terms which will prove to be important later. The diagrams are identical to the Feynman diagrams of covariant perturbation theory, in fact, it is plausible from Eq. (8) that the iterative solution of the tower of equations for the coefficient functions in the Volterra expansion produces all the connected Feynman diagrams of the covariant theory. However, the mathematical expressions associated with these diagrams are definitely different from covariant perturbation theory, and it is in fact not clear whether a set of rules analogous to the Feynman rules can be given that allow to read off the mathematical expressions directly from the diagrams.

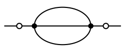
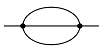
We have now obtained an expression for the vacuum wave functional $\psi(\phi)$, albeit in the form of a perturbative series, that we can use in the functional integral representation (1) of the equal-time correlation functions. Writing $|\psi(\phi)|^2 = \exp(-S')$, we can read off the counterpart S' of the (Euclidean) classical action from Eq. (2). Let us suppose that $\psi(\phi)$ takes real values, then the coefficient functions in a Volterra expansion of S' are given directly by $2f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k})$. For a perturbative evaluation of the equal-time correlation functions we can hence follow the procedures of common covariant perturbation theory. In particular, we can use Feynman diagrams with "bare propagators" $(2\omega_{\mathbf{p}})^{-1}$ and vertices $-2(f_2(\mathbf{p})-\omega_{\mathbf{p}})$ and $-2f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k})$ for $2k \geq 4$. The fact that there is an infinite number of vertices has an important consequence for a first intent to formulate Dyson-Schwinger equations for the equal-time correlation functions: each equation in this infinite tower of equations would have an infinite number of terms by itself, which would make it very difficult to find sensible approximations in a nonperturbative regime like the deep infrared regime of QCD.

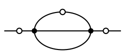
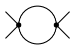
Each of the vertex functions $-2f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k})$ is an infinite power series in λ. Nevertheless, we can use the results obtained before for a perturbative determination of the equal-time correlation functions. Then to a given n-point equal-time correlation function to a fixed order λ^ℓ there are contributions from several vertex functions to different orders. The diagrammatic representation of the correlation functions in terms of the "propagator" and the vertices coming from the Volterra expansion of the (exponent of the) vacuum wave functional can be merged with the diagrammatic representation of the coefficients of this latter expansion. As a result, we get several different contributions in this merged representation that correspond to one and the same Feynman diagram in covariant perturbation theory.

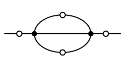
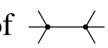
This is most easily seen in an example: let us represent the "propagator" $(2\omega_{\mathbf{p}})^{-1}$ as and use the diagrammatic representation of the "vertices" $-2f_{2k}(\mathbf{p}_1,\ldots,\mathbf{p}_{2k})$ described before (disregarding in the diagrams the factors of two that arise from the square of the vacuum functional). We list all the contributions to the 2-point equal-time correlation function with the topology

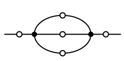
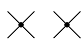


in the merged diagrammatic representation:

1.  = contraction of  from $f_2(\mathbf{p}_1,\mathbf{p}_2)$ with

2.  = contraction of  from $f_4(\mathbf{p}_1,\ldots,\mathbf{p}_4)$ with

3.  = contraction of  from $f_6(\mathbf{p}_1,\ldots,\mathbf{p}_6)$ with

4.  = contraction of  from $f_4(\mathbf{p}_1,\ldots,\mathbf{p}_4)$ with

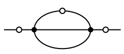
Now, there is a different way of calculating equal-time correlation functions, which is by projecting the usual covariant correlation functions to equal times by integrating over the energy variables in the momentum representation,

$$\langle \phi(\mathbf{p}_1, t=0) \cdots \phi(\mathbf{p}_n, t=0) \rangle = \int \frac{dp_1^0}{2\pi} \cdots \frac{dp_n^0}{2\pi} \langle \phi(\mathbf{p}_1, p_1^0) \cdots \phi(\mathbf{p}_n, p_n^0) \rangle \,. \tag{9}$$

As might have been expected, the projection of a given covariant Feynman diagram to equal times gives the same result as the sum of all diagrams in the merged representation with the same topology as the covariant diagram. We have actually used the representation (9) to check the results obtained in the way described before, explicitly for $\lambda \phi^4$ theory up to two-loop order for the 2-point function and to one-loop order for the 4-point function. It is important to mention that the actual determination of the mathematical expressions for the n-point equal-time correlation functions for $n \geq 4$ is *much* simpler in the way we have described here than by using Eq. (9).

We have also calculated the gluon and ghost 2-point equal-time correlation functions to one-loop order in Yang-Mills theory in Coulomb gauge in both ways, via the solution of the Schrödinger equation with the Christ-Lee Hamiltonian to determine $\psi(\mathbf{A})$ to the corresponding order, and from the equal-time projection of the four-dimensional correlation functions obtained recently in Ref. [7].

In the course of the calculations we have observed that the sum of diagrams with the same topology in the merged representation leads to expressions that are remarkably simpler than the expressions corresponding to the individual diagrams themselves. A typical example would be

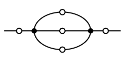


$$= \frac{1}{2} \frac{(-2\lambda)^2}{(2\omega_{\mathbf{p}})^2} \int \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{q}'}} \frac{1}{2\omega_{\mathbf{p}} + 2\omega_{\mathbf{q}'}} \times \frac{1}{2\omega_{\mathbf{p}} + 2\omega_{\mathbf{q}} + 2\omega_{\mathbf{q}'}} \frac{1}{(\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \omega_{\mathbf{q}'} + \omega_{\mathbf{p}+\mathbf{q}+\mathbf{q}'})^2} \tag{10}$$

($\mathbf{p}$ is the external momentum), to be compared with the result for the sum of all diagrams with this topology,

$$\frac{1}{3!} \frac{(-2\lambda)^2}{(2\omega_{\mathbf{p}})^3} \int \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \frac{2\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \omega_{\mathbf{q}'} + \omega_{\mathbf{p}+\mathbf{q}+\mathbf{q}'}}{2\omega_{\mathbf{q}} 2\omega_{\mathbf{q}'} 2\omega_{\mathbf{p}+\mathbf{q}+\mathbf{q}'} (\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \omega_{\mathbf{q}'} + \omega_{\mathbf{p}+\mathbf{q}+\mathbf{q}'})^2} \,. \tag{11}$$

In the latter expression, all the factors appearing in the denominator can be attributed to the propagators and the elementary vertices [cf. Eq. (7)], except for one extra factor $2\omega_{\mathbf{p}}$, which is evidently not the case for Eq. (10). What is more, the expression (11) is very simply related to



$$= \frac{1}{3!} \frac{(-2\lambda)^2}{(2\omega_{\mathbf{p}})^2} \int \frac{d^3q}{(2\pi)^3} \frac{d^3q'}{(2\pi)^3} \times \frac{1}{2\omega_{\mathbf{q}} 2\omega_{\mathbf{q}'} 2\omega_{\mathbf{p}+\mathbf{q}+\mathbf{q}'} (\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \omega_{\mathbf{q}'} + \omega_{\mathbf{p}+\mathbf{q}+\mathbf{q}'})^2} \,, \tag{12}$$

and the latter is obtained directly from the Feynman rules with the "bare propagator" and the elementary vertex only. We will term the latter type of diagram an "F-diagram", and the sum of all diagrams with the same topology an "E-diagram".

The expression (11) can be obtained from (12) by multiplying with $\omega_{\mathbf{k}}$ for every propagator (with momentum $\mathbf{k}$) in the diagram and adding, and then dividing by a similar sum of $\omega_{\mathbf{k}}$ restricted to the external propagators of the diagram. We call this formal procedure of turning an F-diagram into an E-diagram the "E-operator". It has been found to work for all diagrams considered so far, even in Coulomb gauge Yang-Mills theory, in the following sense: to find the contribution to an n-point equal-time correlation function from all diagrams in the merged representation with a given topology, write down the corresponding F-diagram and apply the E-operator to all factors with independent momentum flows. If this tentative rule should turn out to be correct to all orders, we will have found a method to write down the contributions to a given order in λ to any equal-time correlation function immediately, analogously to the Feynman rules for covariant correlation functions. We can even hope to establish Dyson-Schwinger-type equations (with a finite number of terms in every equation) for the equal-time correlation functions, but they will most probably involve fictitious n-point correlation functions corresponding to the sum of all F-diagrams with n external legs.

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