Analytical Description of Neutrino Oscillations in the Earth

A.D. Supanitsky, J.D. D'Olivo and G. Medina-Tanco

Departamento de Física de Altas Energías, Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, A. P. 70-543, 04510, México, D. F., México.

Abstract. We present an analytical description of neutrino oscillations in matter based on the Magnus expansion of the time evolution operator. This approach incorporates in a simple and, at the same time, accurate way the Earth matter effects on the flavor transition probabilities in a wide interval of the neutrino energies. As a concrete application, we examine the daily change in the observed flux of the electron neutrinos coming from the sun.

Keywords: Neutrinos, Matter Effects, Magnus Expansion

INTRODUCTION

The determination of the angle θ_{13} and the CP-violating phase in the leptonic mixing matrix, as well as the determination of the neutrino mass hierarchy, will be the main goals of the next generation of neutrino oscillations experiments. In turn, the interpretation of the forthcoming results will require more careful theoretical descriptions of neutrino oscillations that incorporate sub-leading processes. A subject of particular interest within this context, refers to the matter effects on the flavor transformations for neutrinos propagating through the Earth. The problem has been investigated by direct numerical integration of the equation that governs flavor evolution in a medium. Besides, analytic calculations have been implemented to simplify the numerical computations and to gain a better understanding of the underlying physics. For a varying density these studies have been developed on the basis of the perturbation theory, both in the low [1] and high [2] energy regimes. In this work, we present a novel analytic description of the effect based on the Magnus exponential expansion of the time-displacement operator $\mathscr{U}(t,t_0)$ [3]. This approach incorporates in a simple way the Earth matter effects on the transition probabilities for neutrinos within a wide interval of energies and, in the case of solar neutrinos, it makes possible a accurate description of regeneration phenomenon.

FORMALISM

The evolution of the flavor amplitudes of a neutrino system is conveniently described in terms of the operator $\mathscr{U}(t,t_0)$, which satisfies the Schrödinger-like equation [4] ($\hbar = c = 1$)

$$i\,\frac{d\mathscr{U}}{dt}(t,t_0) = H(t)\,\mathscr{U}(t,t_0), \tag{1}$$

with the initial condition $\mathscr{U}(t_0,t_0) = I$.

Typically, the quantity of interest is the probability P_{ν_e} of observing an electron neutrino at a distance $L \simeq t_f - t_0$ from a source. If $|\nu(t_f)\rangle$ represents the neutrino state at time t_f, then $P_{\nu_e} = |\langle \nu_e|\nu(t_f)\rangle|^2 = |\langle \nu_e|\mathscr{U}(t_f,t_0)|\nu(t_0)\rangle|^2$, where $|\nu(t_0)\rangle$ denotes a certain initial state. We consider oscillations between two neutrino flavors, let say ν_e and ν_a. In the relativistic limit and after discarding an overall phase, the Hamiltonian of the system in the flavor basis $\{|\nu_e\rangle, |\nu_a\rangle\}$ can be written as

$$H(t) = \frac{\Delta_0}{2}\begin{pmatrix} -\cos 2\theta & \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} + \frac{V(t)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

where θ is the mixing angle in vacuum and we have defined $\Delta_0 \equiv \delta m^2/2E$, with E the neutrino energy and δm^2 the squared mass difference. The effect of the medium is accounted for by means of $V(t) = V_e(t) - V_a(t)$, the difference of the potential energies for ν_e and ν_a. To lowest order in the Fermi constant G_F, in normal matter $V(t) = \sqrt{2}\, G_F n_e(t)$, where $n_e(t)$ is the number density of electrons along the neutrino path.

The evolution operator in the flavor basis can be expressed as $\mathscr{U}(t_f,t_0) = U_m(t_f)\,\mathscr{U}^{\mathscr{A}}(t_f,t_0)\,U_m^\dagger(t_0)$, in terms of the corresponding operator $\mathscr{U}^{\mathscr{A}}(t,t_0)$ in the adiabatic basis of the (instantaneous) eigenstates $\{|\nu_{1m}(t)\rangle, |\nu_{2m}(t)\rangle\}$ of $H(t)$. Here, $U_m(t) \equiv U(\theta_m(t))$ is the orthogonal 2×2 matrix that, at each time, diagonalizes the matrix in Eq. (2). The mixing angle in matter $\theta_m(t)$ is given by $\sin 2\theta_m(t) = \Delta_0 \sin 2\theta/\Delta_m(t)$, where $\Delta_m(t) = \Delta_0\,[(\varepsilon(t) - \cos 2\theta)^2 + \sin^2 2\theta]^{1/2}$ stands for the difference between the energy eigenvalues expressed in terms the non-dimensional quantity $\varepsilon(t) = V(t)/\Delta_0 = 2EV(t)/\delta m^2$.

If $V(t)$ is symmetric with respect to the middle point of the neutrino trajectory $\bar{t} = (t_f + t_0)/2$, then $\theta_m(t_f) = \theta_m(t_0) \equiv \theta_m^0$ and $\mathscr{U}(t_f,t_0) = U_m(t_0)\,\mathscr{U}^{\mathscr{A}}(t_f,t_0)\,U_m^\dagger(t_0)$. This is the situation for the Earth, in which case θ_m^0 is the angle evaluated at the surface. In what follows, we restrict ourselves to such a case and find an analytical approximation for $\mathscr{U}(t_f,t_0)$ in terms of $\mathscr{U}^{\mathscr{A}}(t_f,t_0)$ calculated by means of the first two terms in the expansion of the Magnus operator.

Proceeding in this manner, after some algebraic manipulations we arrive at

$$\mathscr{U}^{\mathscr{A}}(t_f,t_0) \cong \begin{pmatrix} \left(\cos\xi - i\sin\xi\,\frac{\xi_{(2)}}{\xi}\right) e^{i\phi_{\bar{t}\to t_f}} & i\sin\xi\,\frac{\xi_{(1)}}{\xi} \\ i\sin\xi\,\frac{\xi_{(1)}}{\xi} & \left(\cos\xi + i\sin\xi\,\frac{\xi_{(2)}}{\xi}\right) e^{-i\phi_{\bar{t}\to t_f}} \end{pmatrix}, \tag{3}$$

where $\phi_{x\to y} = \int_x^y dt'\,\Delta_m(t')$, $\xi = \sqrt{\xi_{(1)}^2 + \xi_{(2)}}$, $\xi_{(1)} = 2\int_{\bar{t}}^{t_f} dt'\;\dot{\theta}_m(t')\,\sin\phi_{\bar{t}\to t'}$, and $\xi_{(2)} = \int_{t_0}^{t_f} dt' \int_{t_0}^{t'} dt''\;\dot{\theta}_m(t')\dot{\theta}_m(t'')\,\sin\phi_{t'\to t''}$.

Suppose that $|\nu(t_0)\rangle = \alpha|\nu_e\rangle + \beta|\nu_a\rangle$, with α and β non-negative (real) numbers satisfying $\alpha^2+\beta^2 = 1$, then

$$P_{\nu_e} = \alpha^2 + (\beta^2 - \alpha^2)(\mathrm{Im}\,\mathscr{U}_{ea})^2 + 2\alpha\beta\,(\mathrm{Im}\,\mathscr{U}_{ee})\,(\mathrm{Im}\,\mathscr{U}_{ea}), \tag{4}$$

with $\mathrm{Im}\,\mathscr{U}_{ee} = \cos 2\theta_m^0\,\mathrm{Im}\,\mathscr{U}_{11}^{\mathscr{A}} + \sin 2\theta_m^0\,\mathrm{Im}\,\mathscr{U}_{12}^{\mathscr{A}}$ and $\mathrm{Im}\,\mathscr{U}_{ea} = -\sin 2\theta_m^0\,\mathrm{Im}\,\mathscr{U}_{11}^{\mathscr{A}} + \cos 2\theta_m^0\,\mathrm{Im}\,\mathscr{U}_{12}^{\mathscr{A}}$, where, according to Eq. (3), $\mathrm{Im}\,\mathscr{U}_{12}^{\mathscr{A}} = \sin\xi\,\xi_{(1)}/\xi$ and $\mathrm{Im}\,\mathscr{U}_{11}^{\mathscr{A}} =$

$\cos\xi \sin\phi_{\bar{t}\to t_f} - \sin\xi\, \xi_{(2)}/\xi \cos\phi_{\bar{t}\to t_f}$. As we see, to this order only the imaginary parts of the matrix elements of the evolution operator are relevant to the calculation of P_{ν_e}. Formula (4) represents our main result and, in order to illustrate its usefulness, in the next section we will apply it to the regeneration effect of solar neutrinos when they goes trough the Earth.

DAY-NIGHT NEUTRINO ASYMMETRY

The relevant quantity in connection with the solar neutrinos is the probability for a neutrino born as a ν_e in the interior of the Sun, to remain as a ν_e at the Earth. The oscillation parameters controlling the leading effects are $\theta = \theta_{12}$ and $\delta m^2 = \delta m^2_{12}$. If the phase information is lost, as will typically happen for neutrinos traveling a long distance to the detection point, then according to the LMA-MSW solution the averaged survival probability for the electron neutrinos can be written as $\overline{P}(\nu_e \to \nu_e) = \sin^2\theta + \cos 2\theta \cos^2\theta^0_\odot - \cos 2\theta^0_\odot f_{reg}$ [5], where $\theta^0_\odot$ denotes the matter mixing angle at the production point in the interior or the Sun. The regeneration factor $f_{reg} = P_{2e} - \sin^2\theta$ represents the terrestrial matter effects expressed as the difference between the probability for ν_2 to become ν_e after traversing the Earth $P_{2e} \equiv P(\nu_2 \to \nu_e) = |\langle \nu_e|\mathscr{U}(t_f,t_0)|\nu_2\rangle|^2$ and the same probability in vacuum $|\langle \nu_e|\nu_2\rangle|^2 = \sin^2\theta$.

We determine f_{reg} by calculating P_{2e} in terms of Eq. (4), with $|\nu(t_0)\rangle = |\nu_2\rangle = \sin\theta|\nu_e\rangle + \cos\theta|\nu_\mu\rangle$. Accordingly, we get

$$\begin{aligned} f_{reg} \;&=\; \cos 2\tilde{\theta}^0_m \cos 2\theta^0_m (\mathrm{Im}\,\mathscr{U}^{\mathscr{A}}_{12})^2 + \sin 2\tilde{\theta}^0_m \sin 2\theta^0_m (\mathrm{Im}\,\mathscr{U}^{\mathscr{A}}_{11})^2 \\ &\quad - \sin(2\tilde{\theta}^0_m + 2\theta^0_m)(\mathrm{Im}\,\mathscr{U}^{\mathscr{A}}_{12})(\mathrm{Im}\,\mathscr{U}^{\mathscr{A}}_{12})\,. \end{aligned} \tag{5}$$

Here, $\tilde{\theta}^0_m = \theta^0_m - \theta$ is the rotation angle that relates the basis of the mass eigenstates $\{|\nu_1,\rangle, |\nu_2\rangle\}$ with the adiabatic one, evaluated on the surface of the Earth.

In order to compare our results with those corresponding to the first and second order in the ε-perturbative we consider the so called mantle-core-mantle model for the density inside the Earth [6]. In this simple model the electron density is approximated by a double step function and the radius of the core and the thickness of the mantle are assumed to be half of the Earth's radius $R_\oplus$: $n_e(r)/N_A = 5.95$ cm^{-3} for $r \le R_\oplus/2$ and $n_e(r)/N_A = 2.48$ cm^{-3} for $R_\oplus/2 < r \le R_\oplus$. Following Ref. [7] we introduce the function $\delta(E) = (f^{(appr)}_{reg}(E) - f^{(exact)}_{reg}(E))/\bar{f}_{reg}(E)$, where $f^{(appr)}_{reg}$ is given by a certain (approximated) analytical expression, $f^{(exact)}_{reg}$ is obtained from the exact (numerical) solution and $\bar{f}_{reg}(E) = 1/2\ \varepsilon_0 \sin^2\theta$ is the average regeneration factor evaluated at the surface layer. Essentially, $\delta(E)$ represents the relative error of the approximated expression.

In Fig. 1 we plotted $\delta(E)$ as a function of the energy for neutrinos that cross the Earth through its center, in the energy interval relevant for ^{8}B solar neutrinos. As shown there, the relative errors associated to the Magnus approximations are always smaller than those corresponding to the perturbative calculations. The lowest-order Magnus result, derived by putting $\xi_{(2)} = 0$ in Eq. (5), works even better than the second-order perturbative expressions, reducing the relative error to less than 0.5% within the energy

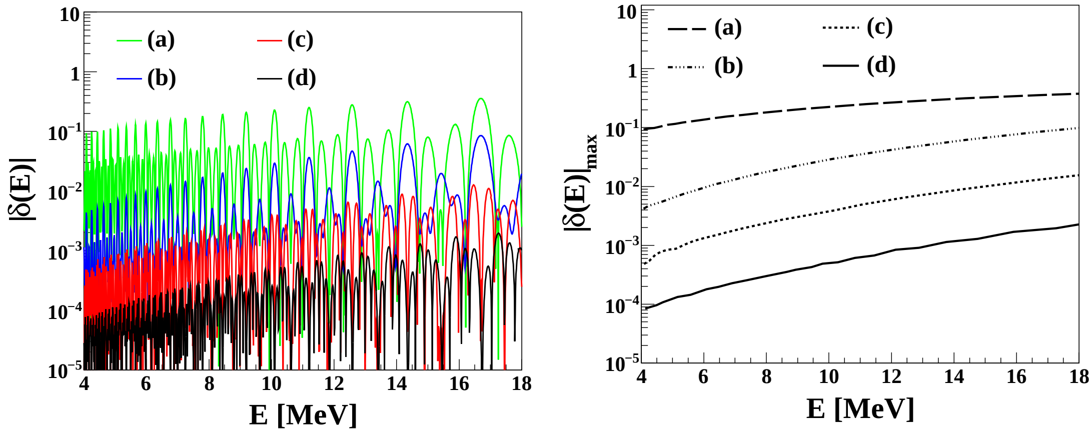

FIGURE 1. The relative error δ as a function of the energy for a neutrino crossing the center of the Earth. Left panel corresponds to the envelopes of $|\delta|$. The oscillation parameters are $\delta m^2_{21} = 8 \times 10^{-5}$ eV^2 and $\tan^2\theta_{12} = 0.4$. (a) and (b) correspond to the first and second order of the perturbative approach, respectively, and (c) and (d) correspond to the first and second order Magnus calculation in the adiabatic basis, respectively.

interval. If the second order term in the Magnus expansion is incorporated, then $\delta(E)$ decrease by almost one order of magnitude. Even though here we restricted us to the low energy regime, the same formalism can be applied to higher energies and, in particular, to analyses matter effects on the oscillations of atmospheric neutrinos [3].

ACKNOWLEDGMENTS

ADS is supported by a postdoctoral grant from the UNAM. All the authors acknowledge the support of PAPIIT-UNAM through grants IN115707 and IN115607 and CONACyT through grant 46999-F.

REFERENCES

1. A. N. Ioannisian and A. Y. Smirnov, *Phys. Rev. Lett.* **93**, 241801 (2004). E. K. Akhmedov, M. A. Tórtola, and J. W. Valle, *JHEP* **057**, 0405 (2004). A. N. Ioannisian, N. A. Kazarian, A. Y. Smirnov and D. Wyler, *Phys. Rev. D* **71**, 033006 (2005).
2. E. Kh. Akhmedov, M. Maltoni, and A. Yu Smirnov, *Phys. Rev. Lett.* **95**, 211801 (2005). B. Brahmachari, S. Choubey, and P. Roy, *Nucl. Phys. B* **671**, 483 (2003). W. Liao, *Phys. Rev. D* **77**, 053002 (2008). J. Arafune, M. Koike, and J. Sato, *Phys. Rev. D* **56**, 3093 (1997).
3. A. D. Supanitsky, J. C. D'Olivo and G. Medina-Tanco, *Phys. Rev. D* **78**, 045024 (2008).
4. J. C. D'Olivo and J. A. Oteo, *Phys. Rev. D* **42**, 256 (1990).
5. A. Y. Smirnov, *arXiv:hep-ph/0702061* (2007).
6. F. D. Stacey, *Physics of the Earth*, Wiley, New York, 1977.
7. A. N. Ioannisian, N. A. Kazarian, A. Y. Smirnov and D. Wyler, *Phys. Rev. D* **71**, 033006 (2005).