An Introduction to High Energy Nuclear Collisions

QCD under extreme conditions

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Lecture II: *What does a nucleus look like at high energy?*

QCD at small x, Renormalization Group, Saturation, Color Glass Condensate

The Regge-Gribov limit

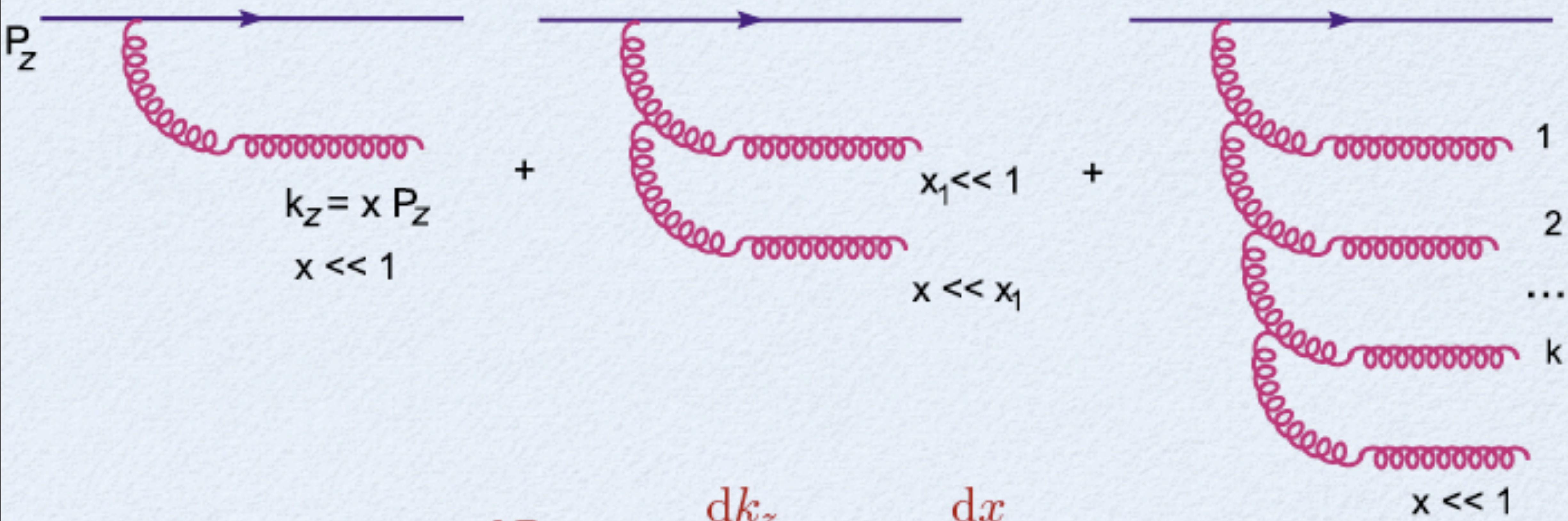
$$\boxed{x_{\mathrm{Bj}} \to 0\,;\; s \to \infty\,;\; Q^2 (>> \Lambda^2_{\mathrm{QCD}}) = \text{fixed}}$$

Physics of strong fields in QCD

Multi-particle production

Novel universal properties of QCD

- The infrared sensitivity of bremsstrahlung favors the emission of 'soft' (= small–x) gluons



$$\mathrm{d}\mathcal{P} \,\propto\, \alpha_s \frac{\mathrm{d}k_z}{k_z} \,=\, \alpha_s \frac{\mathrm{d}x}{x}$$

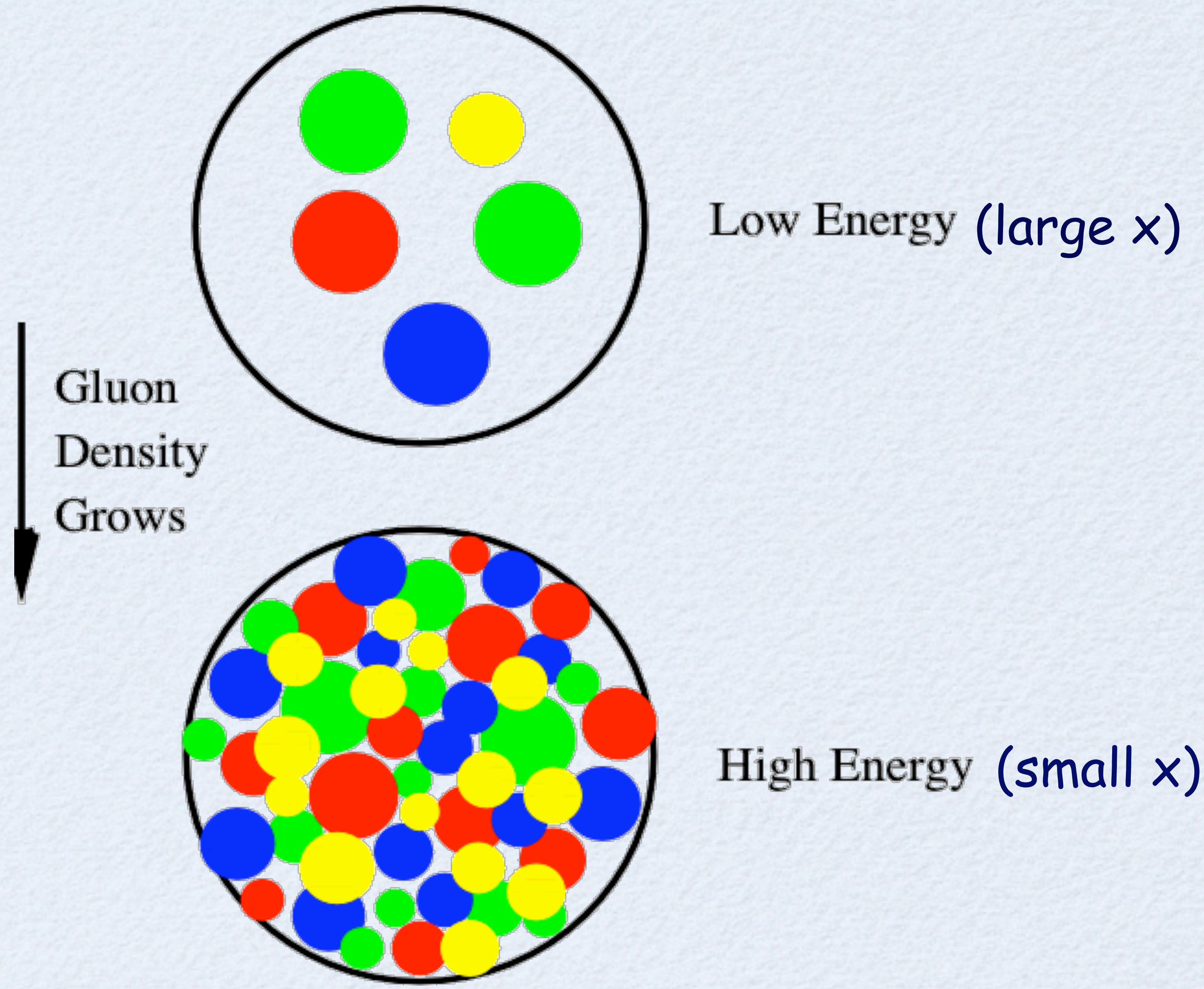
- The 'price' of an additional gluon:

$$\mathcal{P}(1) \,\propto\, \alpha_s \int_x^1 \frac{\mathrm{d}x_1}{x_1} \,=\, \alpha_s \ln \frac{1}{x}$$

number of gluons grows fast $n \sim e^{\alpha_s \, ln 1/x}$

Resolving the hadron

Ren.Group-BFKL evolution
(sums large logs in x)

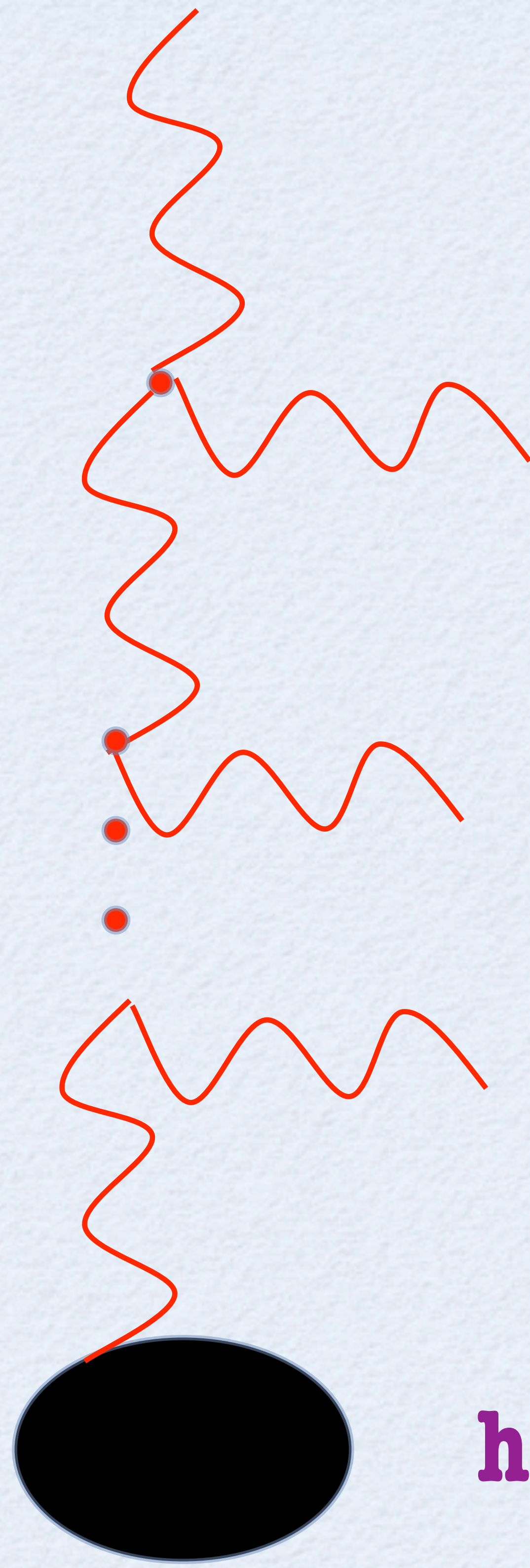


Gluon density saturates at $f = 1/\alpha_s$

- strongest E&M fields in nature...

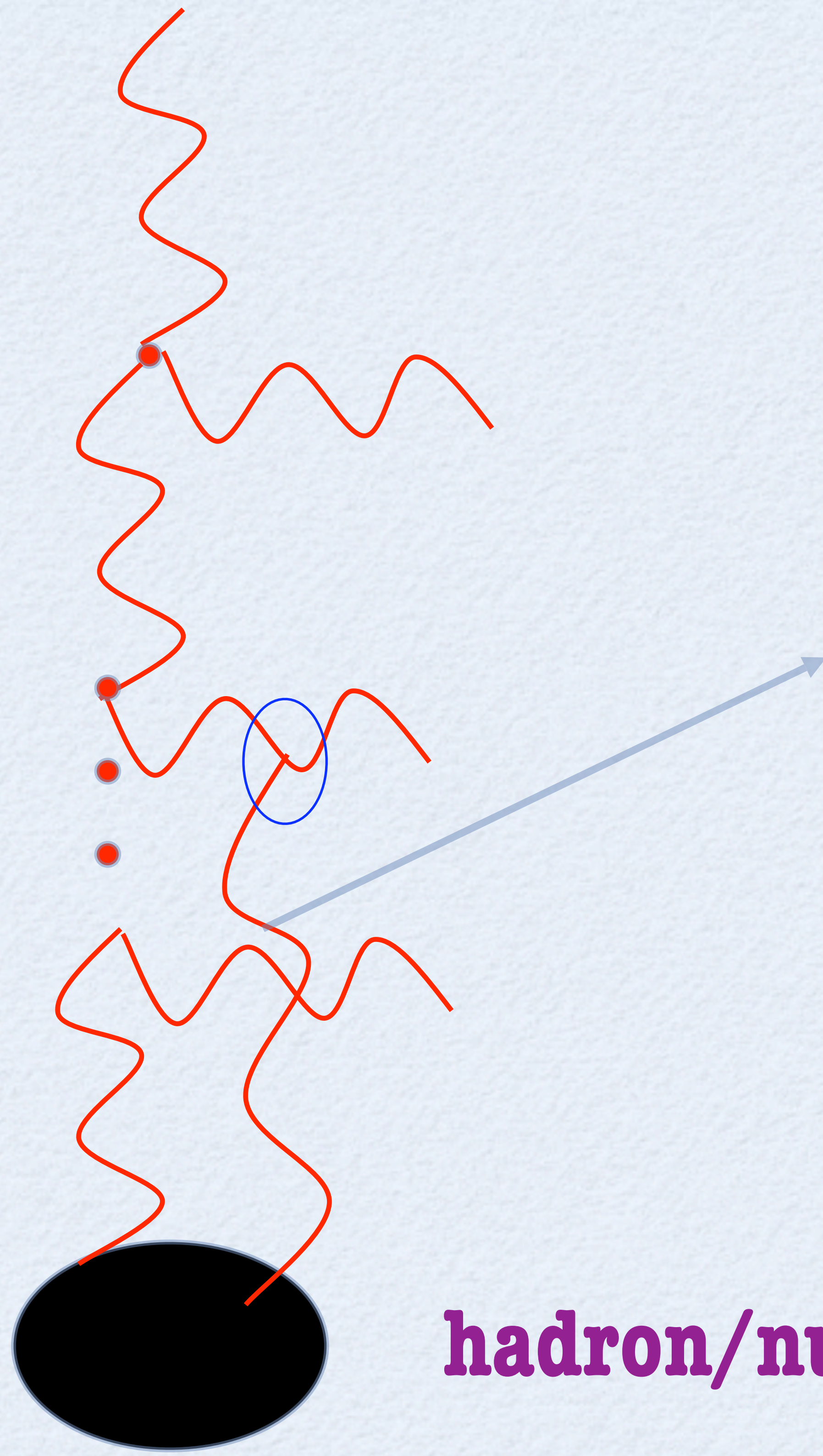
QCD evolution: linear vs. non-linear

QCD Bremsstrahlung



hadron/nucleus

QCD evolution: linear vs. non-linear



Parton Saturation

★Competition between attractive bremsstrahlung and repulsive recombination effects

Maximum occupation number ($f = 1/\alpha_S$) =>

$$\frac{1}{2(N_c^2-1)}\,\frac{x\,G(x,Q^2)}{\pi R^2\,Q^2} = \frac{1}{\alpha_S(Q^2)}$$

This relation is saturated for

$$\boxed{Q = Q_s(x) >> \Lambda_{\rm QCD} \approx 0.2\ \text{GeV}}$$

QCD in high gluon density regime

Need a new organizing principle to explore this novel regime of high energy QCD

Light Cone Coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$a^{1,2}$ unchanged. Notation : $\vec{a}_\perp \equiv (a^1, a^2)$

- Under a Lorentz boost in the z direction :

$$a^+ \to \Lambda\, a^+ \quad , \quad a^- \to \Lambda^{-1}\, a^- \quad , \quad a^{1,2} \to a^{1,2}$$

$$x \equiv \frac{k^+}{P^+}$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{x}_\perp \cdot \vec{y}_\perp$$

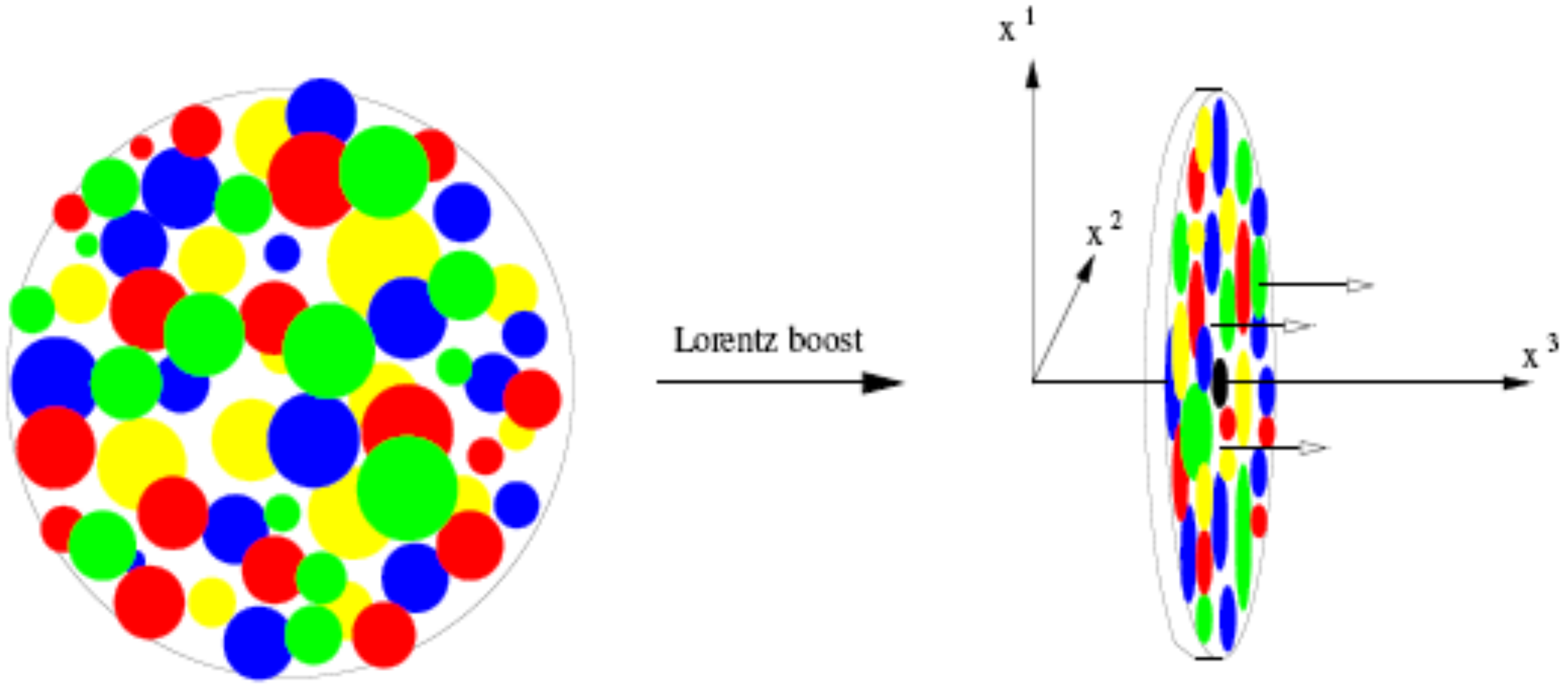
$$d^4x = dx^+ dx^- d^2\vec{x}_\perp$$

$$\Box = 2\partial^+\partial^- - \vec{\nabla}_\perp^2$$

Notation : $\partial^+ \equiv \dfrac{\partial}{\partial x^-}$, $\partial^- \equiv \dfrac{\partial}{\partial x^+}$

What does a nucleus look like in the IMF ?

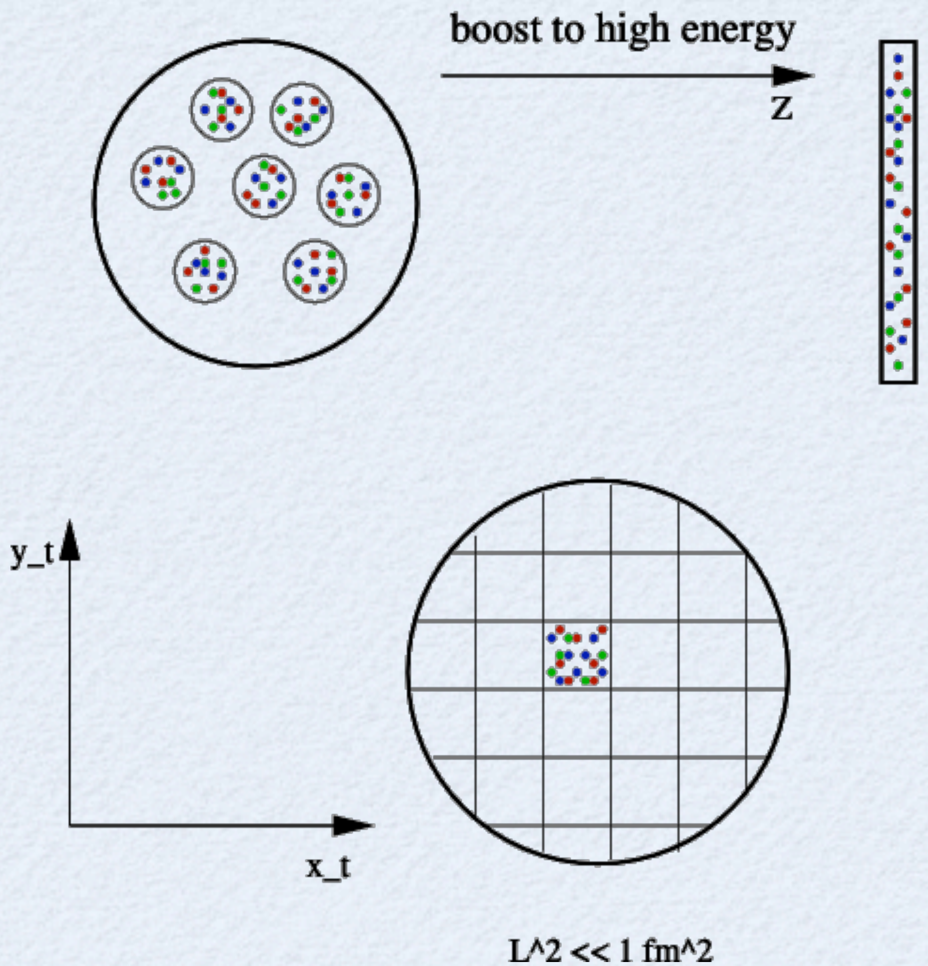
- In ∞-momentum frame, nucleus is a thin sheet of color charge

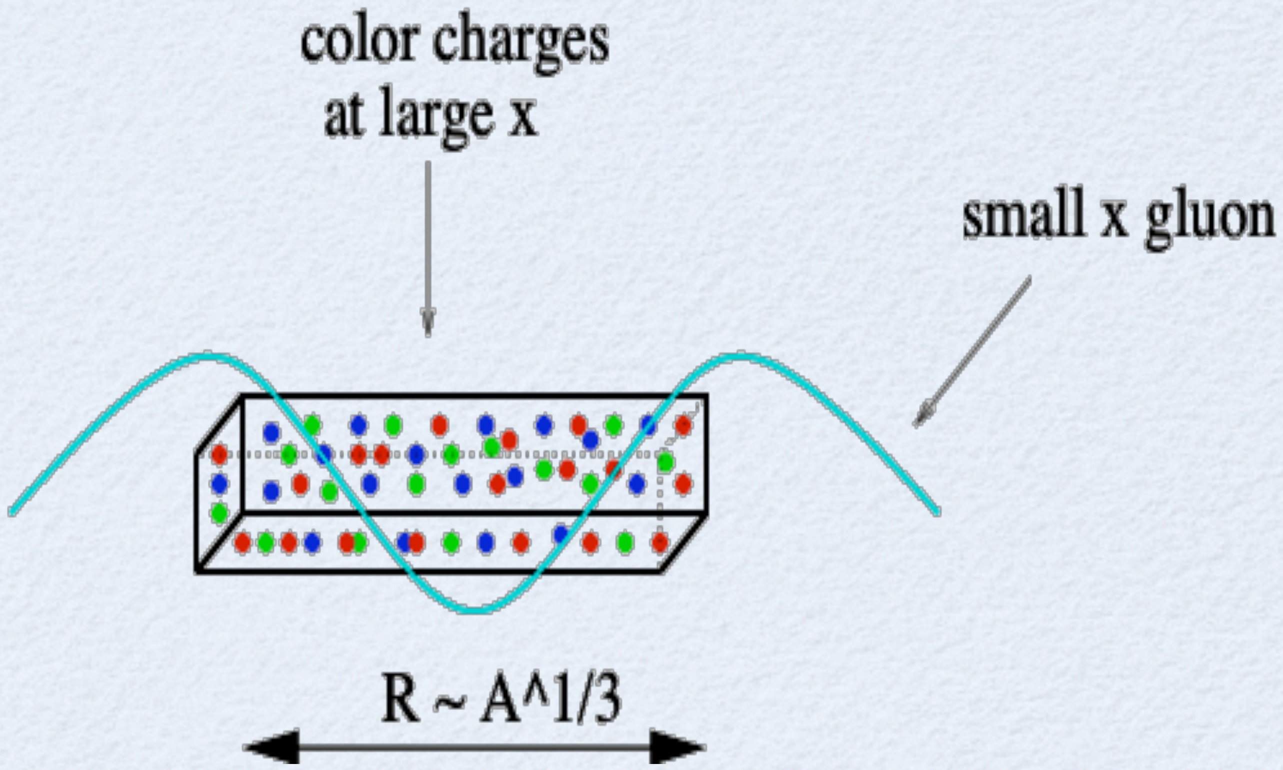


- sheet travels in the x^+ direction with while sitting at $x^- = 0$,

$$J^\mu(x) = \delta^{\mu +}\delta(x^-)\rho(x^1, x^2)$$

What does a nucleus look like in the IMF ?



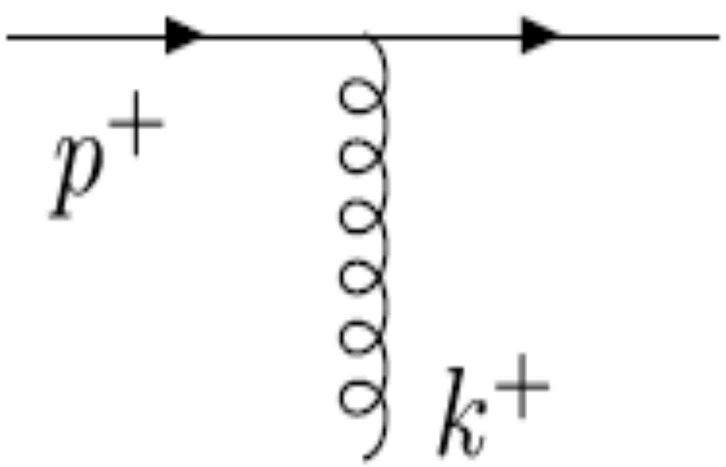


$$\lambda_{\text{wee}} \approx \frac{1}{k^+} \equiv \frac{1}{xP^+} >> \lambda_{\text{val.}} \equiv \frac{Rm_p}{P^+} => x << A^{-1/3}$$

wee partons (small x gluons) "see" a large density of color sources at small transverse resolutions

Time scales

- Take a look at the following radiative proces:



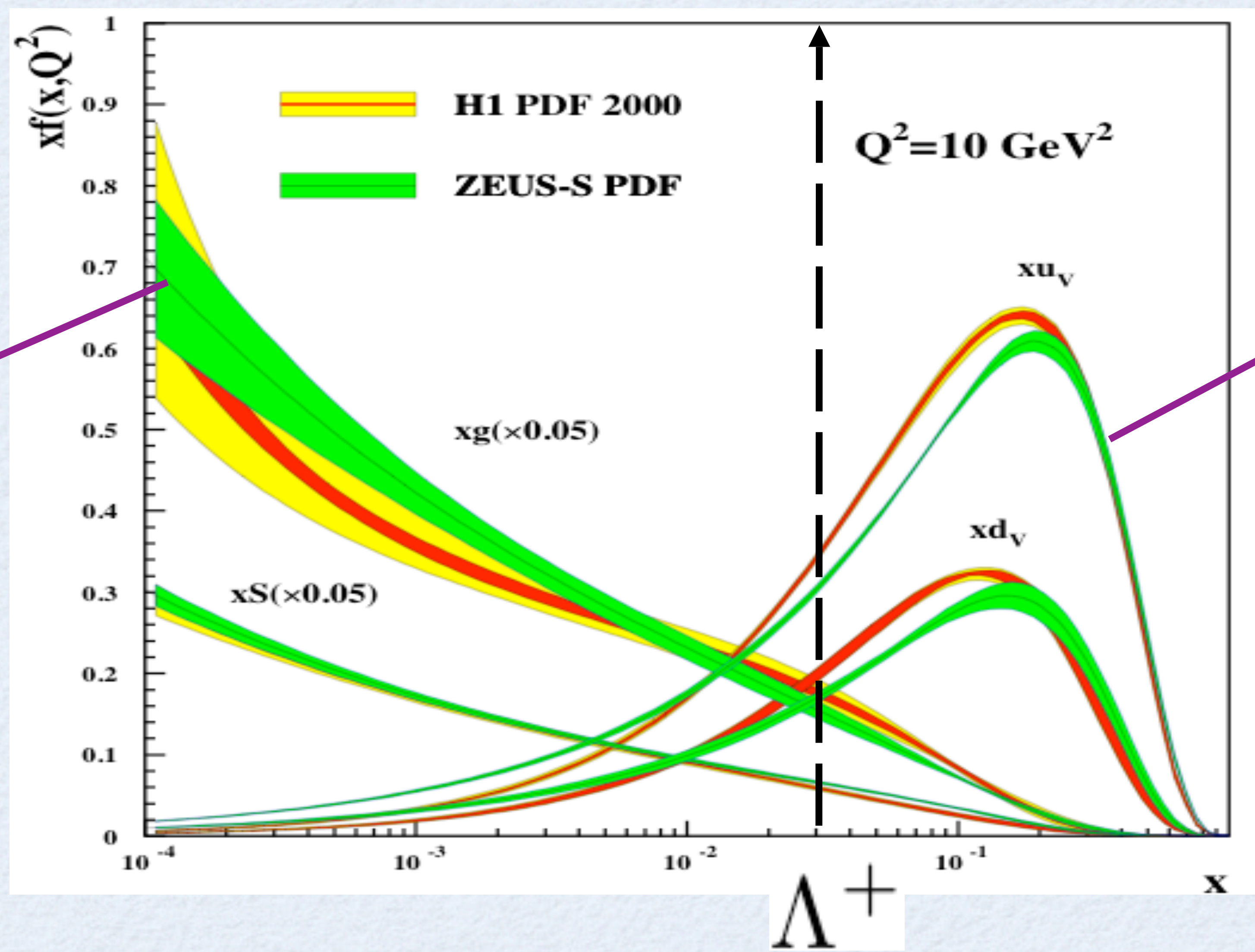
- $k^+ \ll p^+$
- light-cone lifetime of gluon:

$$\Delta x^+ \simeq \frac{1}{k^-} \ll \frac{1}{p^-}$$

- with LC-energies p^-, k^-

- During the short life of the gluon, the dynamics of the fast parton are frozen (remember 'Glass')

Born-Oppenheimer separation of large and small x modes



Valence modes - static sources for wee partons

Dynamical wee modes

$$\tau_{\text{wee}} \sim \frac{1}{k^-} = \frac{2\,k^+}{k_\perp^2} \equiv \frac{2\,xP^+}{k_\perp^2} \qquad \tau_{\text{valence}} = \frac{2P^+}{k_\perp^2} >> \tau_{\text{wee}} \text{ for } x << 1$$

The effective action

Scale separating sources & fields

$$\mathcal{Z}[j] = \int [d\rho]\, W_{\Lambda^+}[\rho] \left\{ \frac{\int^{\Lambda^+} [dA]\, \delta(A^+)\, e^{iS[A,\rho] - \int j\cdot A}}{\int^{\Lambda^+} [dA]\, \delta(A^+)\, e^{iS[A,\rho]}} \right\}$$

Gauge invariant weight functional for distribution of sources

$$S[A,\rho] = \frac{-1}{4} \int d^4x\, F_{\mu\nu}^2 + \frac{i}{N_c} \int d^2x_\perp\, dx^- \delta(x^-) \mathrm{Tr}\left(\rho(x_\perp)\, U_{-\infty,\infty}[A^-]\right)$$

Dynamical wee fields

Coupling of wee fields to sources

$$U_{-\infty,+\infty}[A^-] = \mathcal{P} \exp\left(ig \int dx^+ A^{-,a} T^a \right)$$

This action captures the remarkable properties of hadrons and nuclei at high energies

The large A limit

"Pomeron" excitations "Odderon" excitations

$$W_{\Lambda^+} = \exp\left(-\int d^2x_\perp \left[\frac{\rho^a\rho^a}{2\,\mu_A^2} - \frac{d_{abc}\,\rho^a\rho^b\rho^c}{\kappa_A}\right]\right)$$

$$\mu_A^2 = \frac{g^2 A}{2\pi R^2} \propto A^{1/3} \qquad \kappa_A = \frac{g^3 A^2 N_c}{\pi^2 R^4} \propto A^{2/3}$$

$$\boxed{\mu_A^2 \approx Q_s^2 \;;\; \alpha_S(Q_s^2) << 1}$$

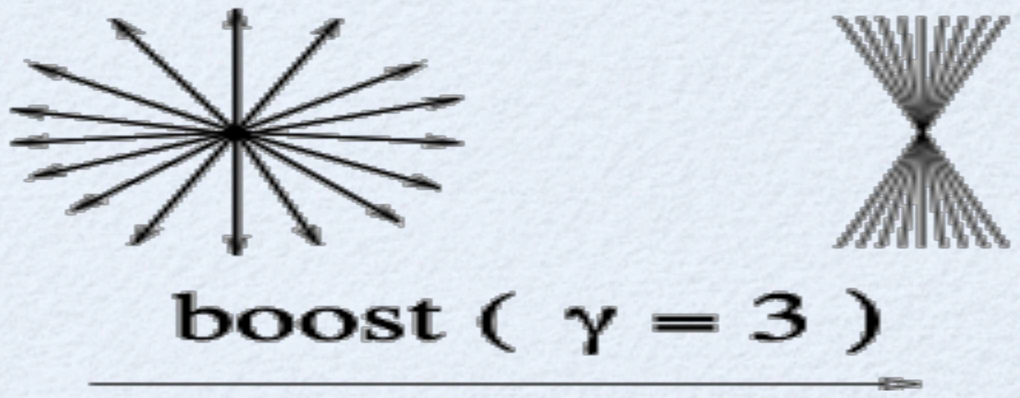
effective action describes a non-perturbative, albeit weakly coupled system with rich dynamics

Classical field of a nucleus

Yang-Mills equations:

$$(D_\mu F^{\mu\nu})^a = J^{\nu,a} \equiv \delta^{\nu +}\,\delta(x^-)\,\rho^a(x_\perp)$$

can be solved exactly:
solutions are non-Abelian
Weizäcker-Williams fields



Saddle point of effective action-> Yang-Mills equations

Solution of Yang-Mills equations

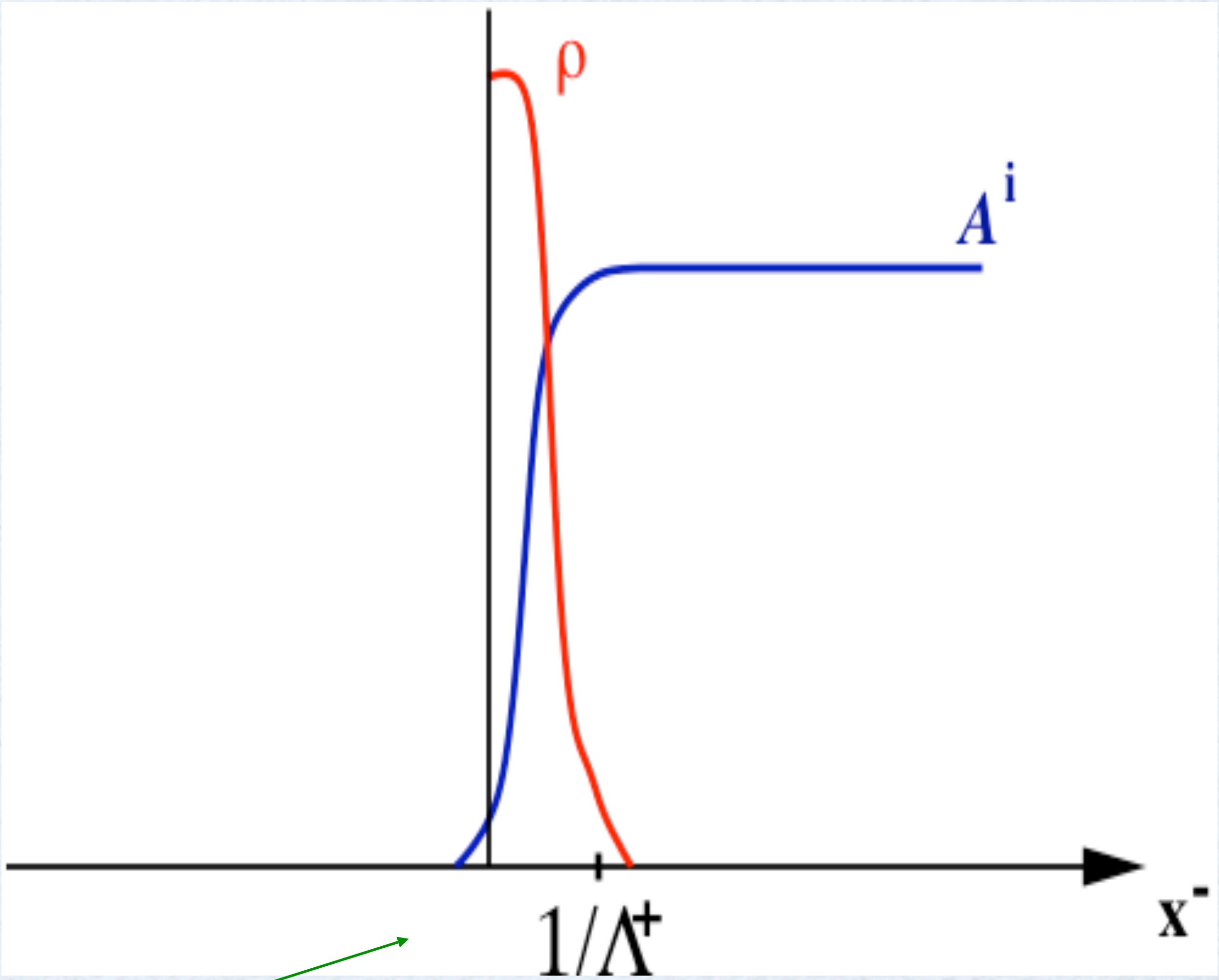
$$A^+ = A^- = 0$$

$$A^i_a = \theta(x^-)\, \alpha^i_a$$

with

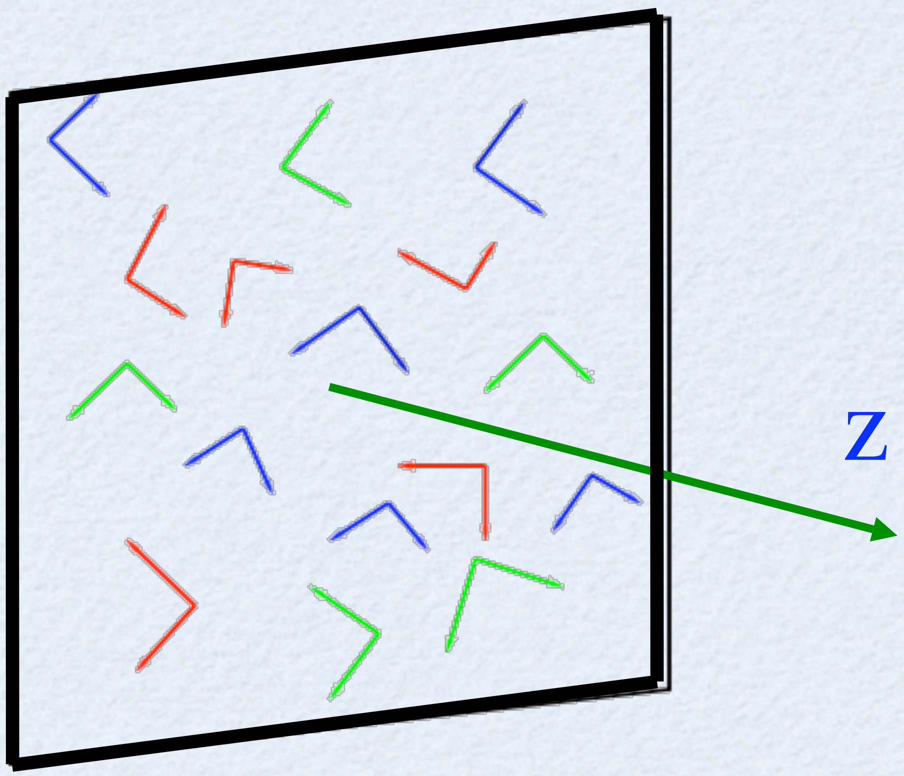
$$\alpha_i \equiv \frac{i}{g} U\, \partial^i\, U^\dagger$$

$$\partial_i\, \alpha_i = g\, \rho$$



careful solution requires smearing in x^-

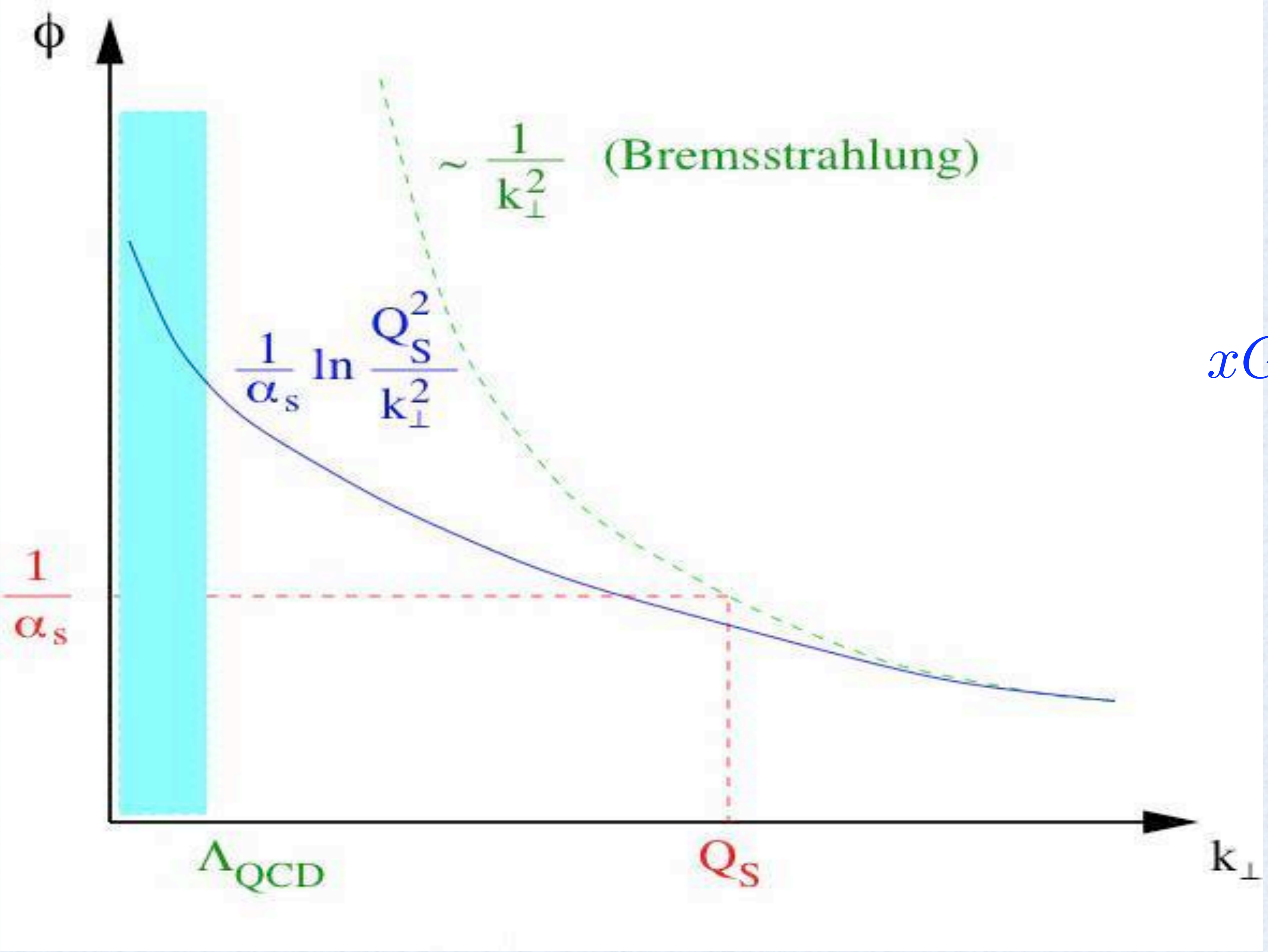
A nucleus at high energy



random electric and magnetic fields
in plane of fast moving nucleus

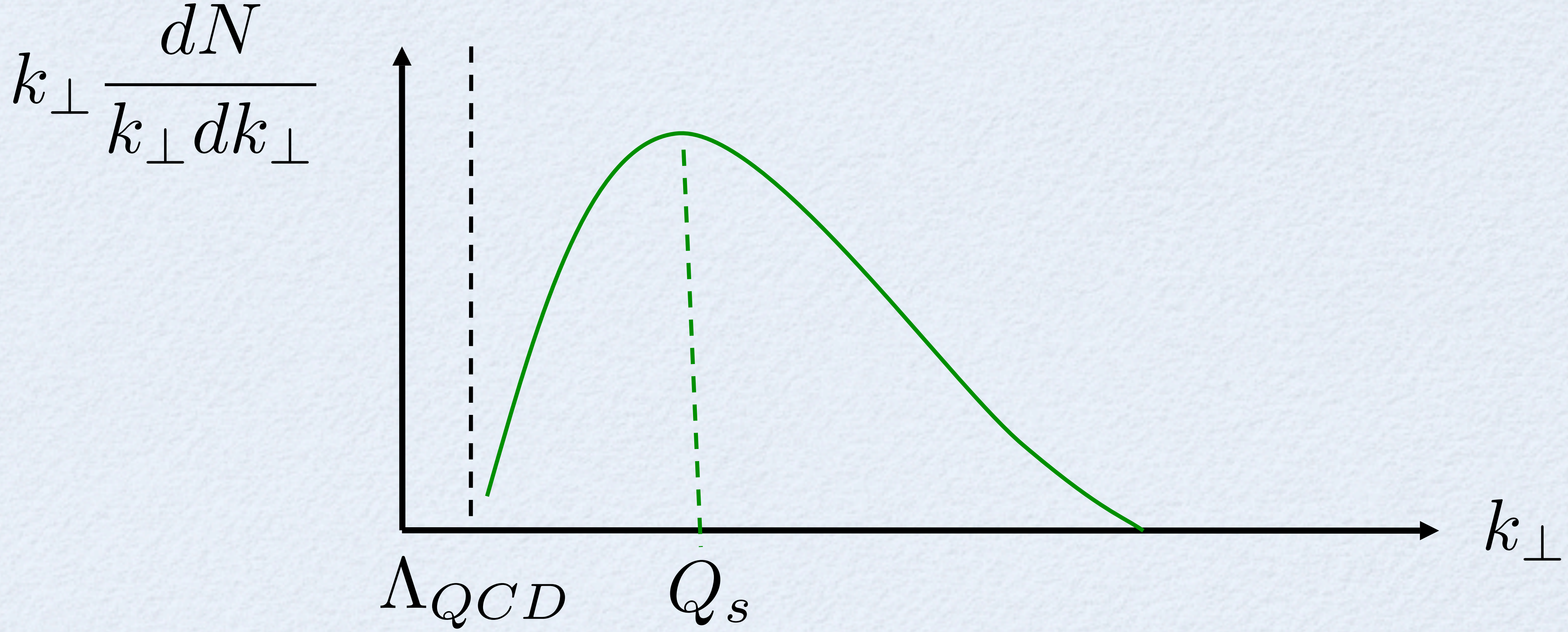
Intrinsic gluon distribution of a nucleus

$$< AA >_\rho = \int [d\rho] A_{\text{cl.}}(\rho) A_{\text{cl.}}(\rho) W_{\Lambda^+}[\rho]$$



$$xG(x, Q^2) \sim \int^{Q^2} d^2 k_t \, \phi(x, k_t)$$

Gluon distribution of a nucleus



most of the gluons in the nucleus have momentum of order of Q_s

High energy limit of QCD

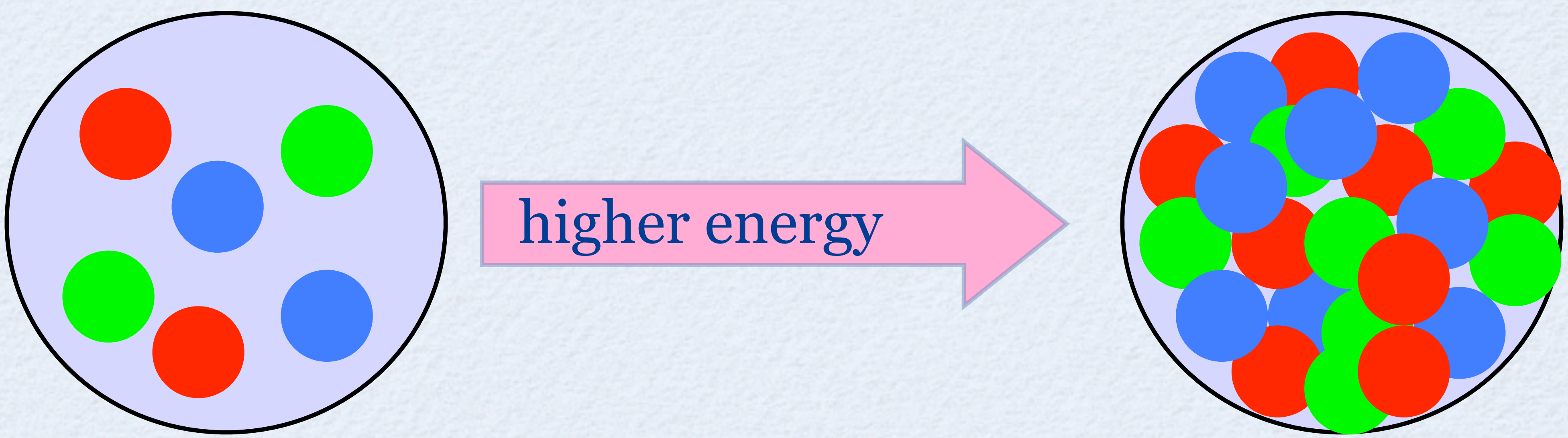
A universal form of matter at high energy

Color Glass Condensate (CGC)

Gluons have "color"

created from "frozen" random color source, that evolves slowly compared to natural time scale

High density ! occupation number $\sim 1/\alpha_s$ at saturation

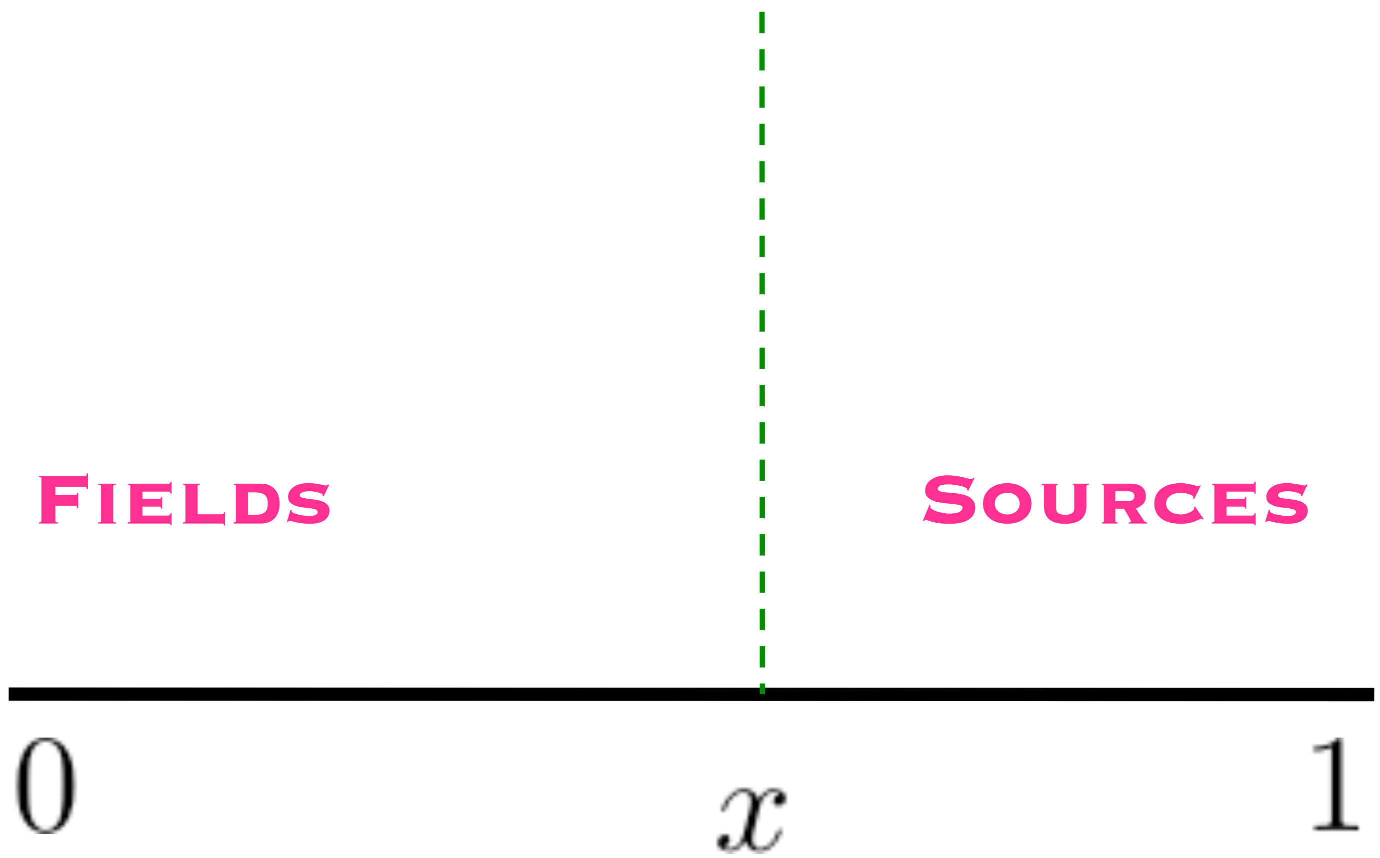


Dilute gas

CGC: high density gluons

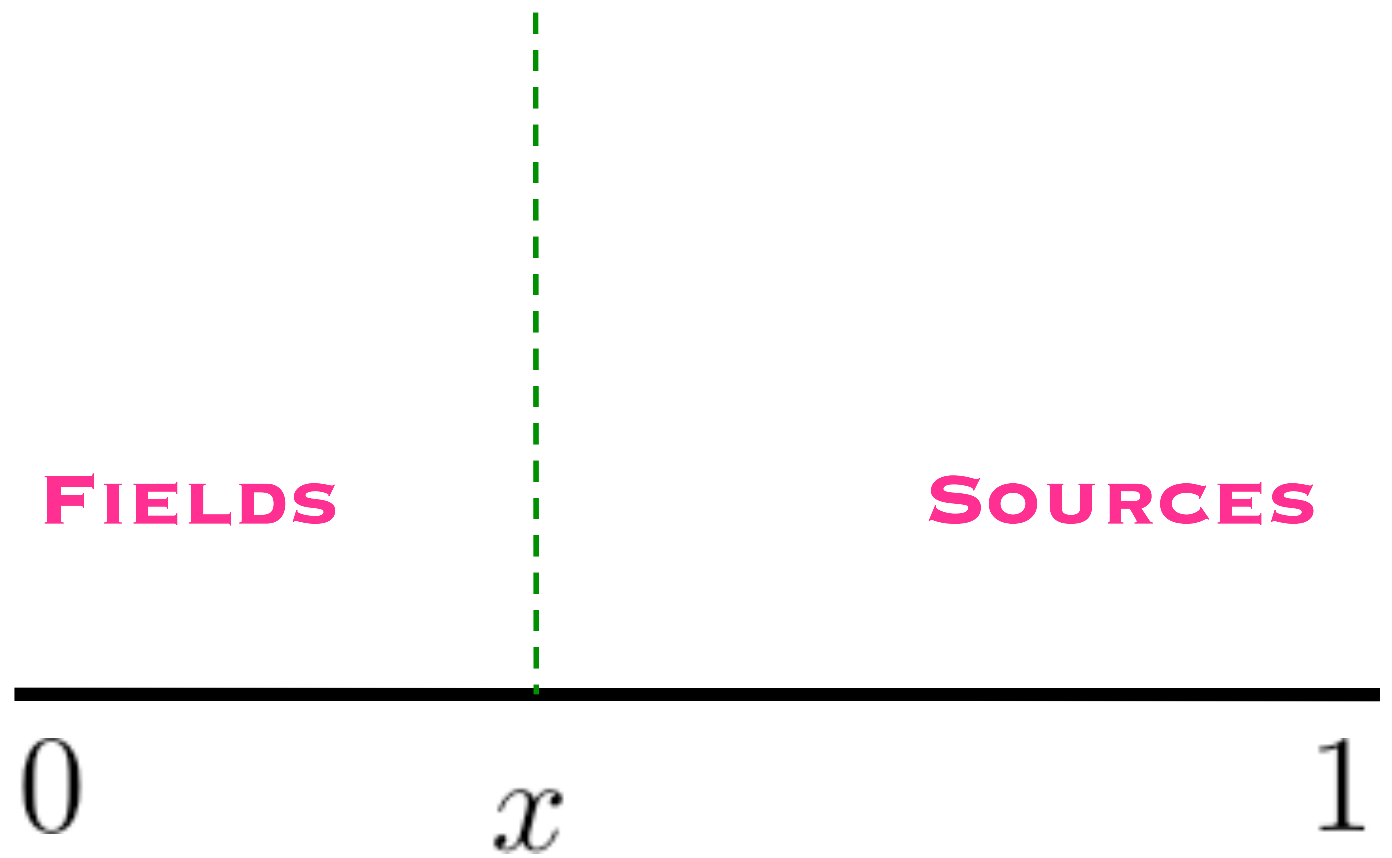
Quantum corrections: Wilsonian RG

(α_s Log 1/x)

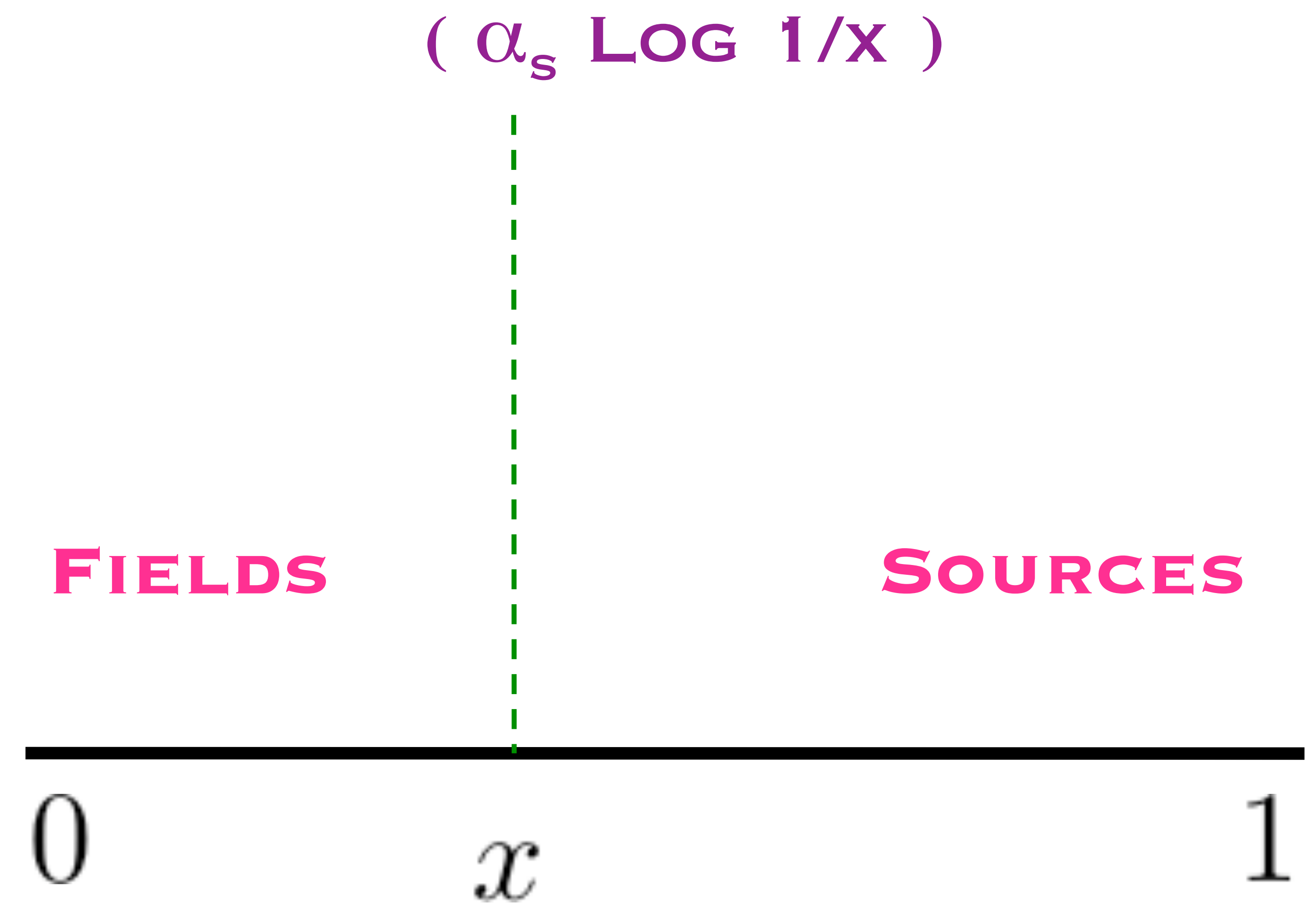


Quantum corrections: Wilsonian RG

(α_s Log 1/x)



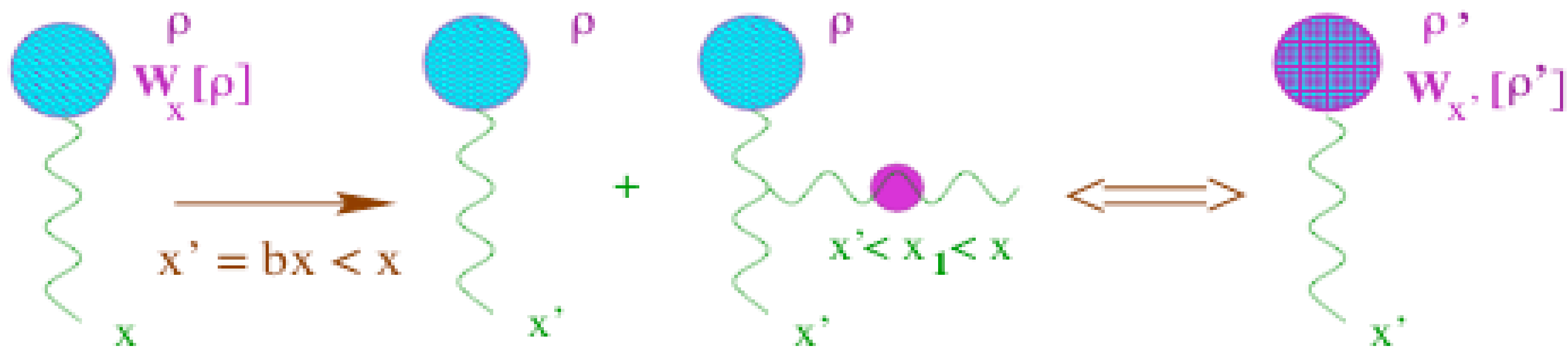
Quantum corrections: Wilsonian RG



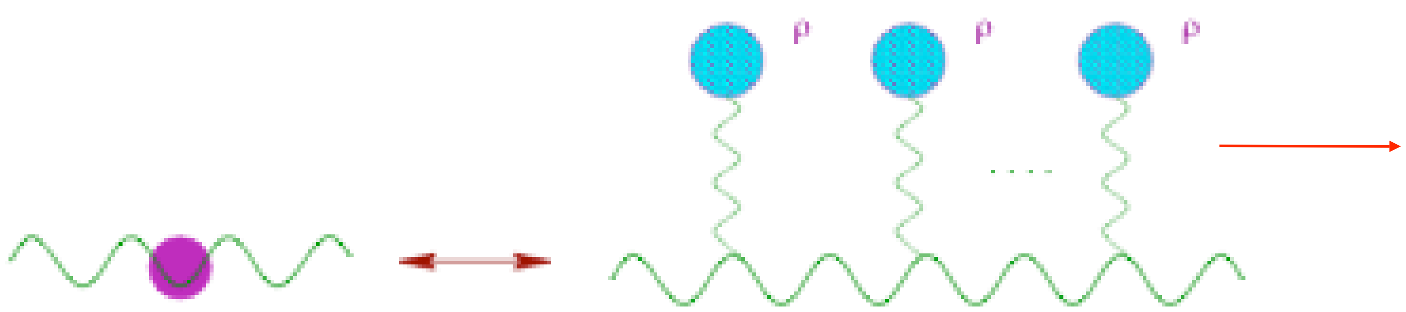
$$\boxed{\frac{\partial W_x[\rho]}{\partial \ln(1/x)} = \frac{1}{2}\int_{x_\perp, y_\perp} \frac{\delta}{\delta \rho_x^a(x_\perp)} \chi^{ab}(x_\perp, y_\perp)[\rho] \frac{\delta}{\delta \rho_x^b(y_\perp)} W_x[\rho]}$$

The JIMWLK (functional RG) equation

Wilson RG at small x



Color charge grows due to inclusion of fields into hard source with decreasing x: $\rho' = \rho + \delta\rho => W_x[\rho] \rightarrow W_{x'}[\rho']$



Because of strong fields $A \sim 1/g$
All insertions are O(1)

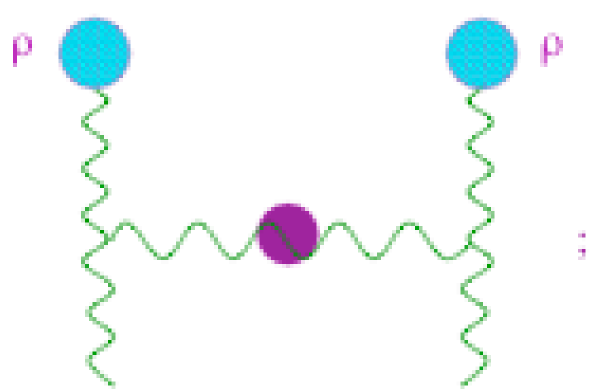
$W_x[\rho]$ *obeys a non-linear Wilson renormalization group equation*

Wilson RG at small x

At each step in the evolution, compute 1-point and 2-point functions in the background field

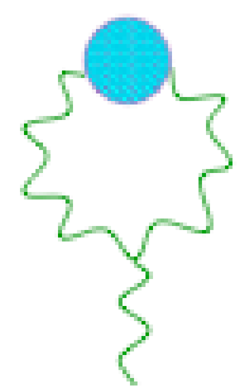
$$\sigma^a(x)[\rho] =< \delta\rho^a_Y(x) >_\rho \; ; \; \chi^{ab}(x,y)[\rho] =< \delta\rho^a_Y(x)\delta\rho^b_Y(y) >_\rho$$

χ =



;

σ =



$$\sigma^a(x) = \frac{1}{2}\int d^2y \frac{\delta\chi^{ab}(x,y)}{\delta\rho^b_Y(y)}$$

$$=> \frac{\partial}{\partial Y} < O[\alpha] >_Y =< \frac{1}{2}\int_{x,y} \frac{\delta}{\delta\alpha^a_Y(x)}\chi^{ab}_{x,y}\frac{\delta}{\delta\alpha^b_Y(y)} O[\alpha] >_Y$$

Consider the 2-point function:
(intrinsic gluon distribution)

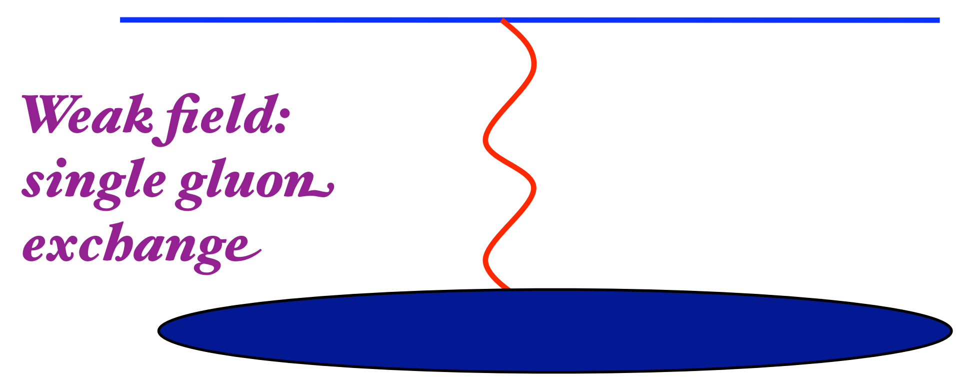
$$< \alpha(x_\perp)\alpha(y_\perp) >_Y$$

Weak field limit: BFKL equation

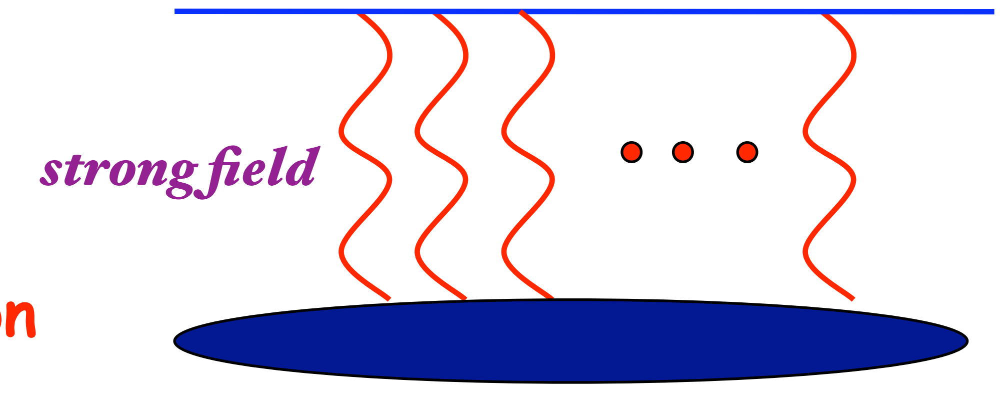
JIMWLK equations describe evolution of all N-point correlation functions with energy

all observables depend on products of U's

$$U(x_t) \equiv \hat{P}\exp[-ig\int dx^- \,\frac{1}{\partial_t^2}\,\rho^a(x^-,x_t)\,T^a]$$



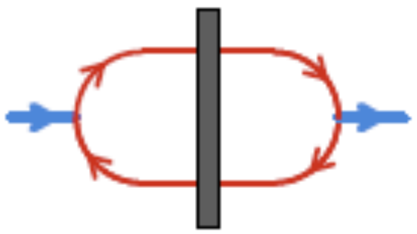
U includes multiple scattering on the target (eikonal propagation)



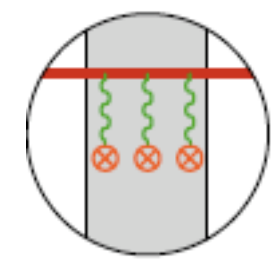
mean field + large N_c : Balitsky-Kovchegov equation

scattering of a quark anti-quark dipole on a target

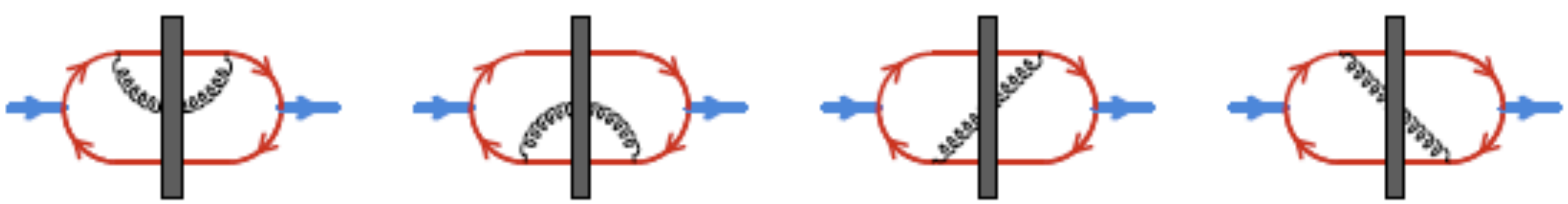
- Assume that the initial and final states α and β are a color singlet $Q\overline{Q}$ dipole. The bare scattering amplitude can be written as :

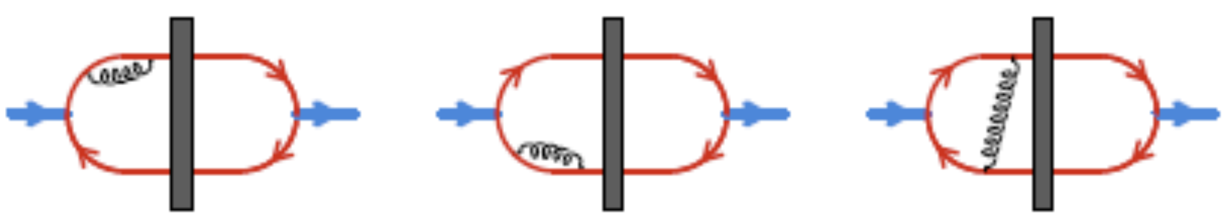


$$\propto \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$



- At one loop, the following diagrams must be evaluated :

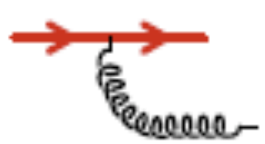




$+\ \mathrm{h.c.}$

the radiation vertex

- In the gauge $A^+ = 0$, the emission of a gluon of momentum k by a quark can be written as :



$$= 2gt^a \, \frac{\vec{\epsilon}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2}$$

- In coordinate space, this reads :

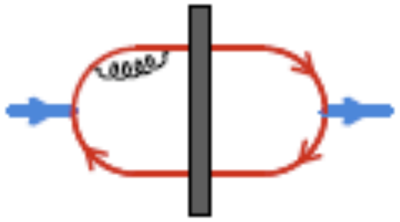
$$\int \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2} \, e^{i\vec{\boldsymbol{k}}_\perp \cdot (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)} \, 2gt^a \, \frac{\vec{\epsilon}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2} = \frac{2ig}{2\pi} t^a \frac{\vec{\epsilon}_\lambda \cdot (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- When connecting two gluons, one must use :

$$\sum_\lambda \vec{\epsilon}_\lambda^{\,i} \vec{\epsilon}_\lambda^{\,j} = -g^{ij}$$

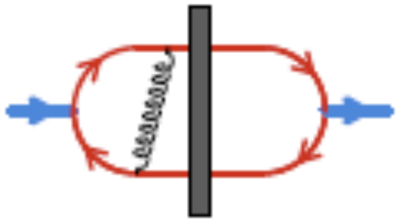
Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole
- Examples :



$$= \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

$$\times -2\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{z}_\perp}{(2\pi)^2} \frac{(\vec{x}_\perp-\vec{z}_\perp)\cdot(\vec{x}_\perp-\vec{z}_\perp)}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{x}_\perp-\vec{z}_\perp)^2}$$



$$= \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)t^a\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{z}_\perp}{(2\pi)^2} \frac{(\vec{x}_\perp-\vec{z}_\perp)\cdot(\vec{y}_\perp-\vec{z}_\perp)}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$

- Reminder : $t^a t^a = (N_c^2-1)/2N_c \equiv C_F$

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F\alpha_s}{\pi^2}\int\frac{dk^+}{k^+}\int d^2\vec{z}_\perp\,\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2\mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

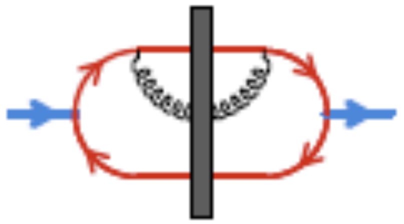
$$\int^{p^+}\frac{dk^+}{k^+}=\ln(p^+)=Y$$

$\triangleright$ When Y is large, $\alpha_s Y$ may not be small. By differentiating with respect to Y, we will get an evolution equation in Y whose solution resums all the powers $(\alpha_s Y)^n$

- Note : the integral over $\vec{z}_\perp$ is divergent when $\vec{z}_\perp=\vec{x}_\perp$ or $\vec{y}_\perp$

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon
- Example :



$$= \left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{x}_\perp) t^b U^\dagger(\vec{y}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{z}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{z}_\perp) \frac{(\vec{x}_\perp - \vec{z}_\perp)\cdot(\vec{x}_\perp - \vec{z}_\perp)}{(\vec{x}_\perp - \vec{z}_\perp)^2(\vec{x}_\perp - \vec{z}_\perp)^2}$$

  - $\widetilde{U}_{ab}(\vec{z}_\perp)$ is a Wilson line in the adjoint representation
- In order to simplify the color structure, first recall that :

$$t^a \widetilde{U}_{ab}(\vec{z}_\perp) = U(\vec{z}_\perp) t^b U^\dagger(\vec{z}_\perp)$$

- Then use the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Real corrections

- The Wilson lines can be rearranged into :

$$\mathrm{tr}\left[t^a U(\vec{x}_\perp) t^b U^\dagger(\vec{y}_\perp)\right] \widetilde{U}_{ab}(\vec{z}_\perp) = \frac{1}{2}\mathrm{tr}\left[U^\dagger(\vec{z}_\perp)U(\vec{x}_\perp)\right]\mathrm{tr}\left[U(\vec{z}_\perp)U^\dagger(\vec{y}_\perp)\right] - \frac{1}{2N_c}\mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

  - The term in $1/2N_c$ cancels against a similar term in the virtual contribution
  - All the real terms have the same color structure

- When we sum all the real terms, we generate the same kernel as in the virtual terms :

$$\frac{(\vec{x}_\perp - \vec{y}_\perp)^2}{(\vec{x}_\perp - \vec{z}_\perp)^2(\vec{y}_\perp - \vec{z}_\perp)^2}$$

- In order to simplify the notations, let us denote :

$$S(\vec{x}_\perp, \vec{y}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

Real + virtual corrections

- The 1-loop scattering amplitude reads :

$$-\frac{\alpha_s N_c^2 Y}{2\pi^2}\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \int d^2\vec{z}_\perp \,\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\left\{S(\vec{x}_\perp,\vec{y}_\perp)-S(\vec{x}_\perp,\vec{z}_\perp)S(\vec{z}_\perp,\vec{y}_\perp)\right\}$$

- Reminder: the bare scattering amplitude was :

$$\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 N_c\, S(\vec{x}_\perp,\vec{y}_\perp)$$

- Hence, we have :

$$\frac{\partial S(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y}=-\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp \,\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\left\{S(\vec{x}_\perp,\vec{y}_\perp)-S(\vec{x}_\perp,\vec{z}_\perp)S(\vec{z}_\perp,\vec{y}_\perp)\right\}$$

  - since $S(\vec{x}_\perp,\vec{x}_\perp)=1$, the integral over $\vec{z}_\perp$ is now regular

The BFKL equation (linear)

Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

- The BFKL equation can be obtained by linearizing the previous equation

- Write $S(\vec{x}_\perp, \vec{y}_\perp) \equiv 1 - T(\vec{x}_\perp, \vec{y}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude T is small. Drop the terms that are non-linear in T :

$$\frac{\partial T(\vec{x}_\perp, \vec{y}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{z}_\perp \, \frac{(\vec{x}_\perp - \vec{y}_\perp)^2}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{y}_\perp - \vec{z}_\perp)^2}$$
$$\times \Big\{ T(\vec{x}_\perp, \vec{z}_\perp) + T(\vec{z}_\perp, \vec{y}_\perp) - T(\vec{x}_\perp, \vec{y}_\perp) \Big\}$$

- The solution of this equation grows exponentially when $Y \to +\infty$ $\triangleright$ serious unitarity problem..., diffusion

$$T \sim e^{\#\,\alpha_s Y} \, exp[-\# \, \frac{ln^2 \frac{k}{k_0}}{Y}]$$

The BK equation (non-linear)

- In fact, the first evolution equation we derived has a bounded solution. The unbounded solutions of BFKL are due to dropping the non-linear term. The full equation reads :

$$\frac{\partial T(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp \, \frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{T(\vec{x}_\perp,\vec{z}_\perp)+T(\vec{z}_\perp,\vec{y}_\perp)-T(\vec{x}_\perp,\vec{y}_\perp)\underline{-T(\vec{x}_\perp,\vec{z}_\perp)T(\vec{z}_\perp,\vec{y}_\perp)}\Big\}$$

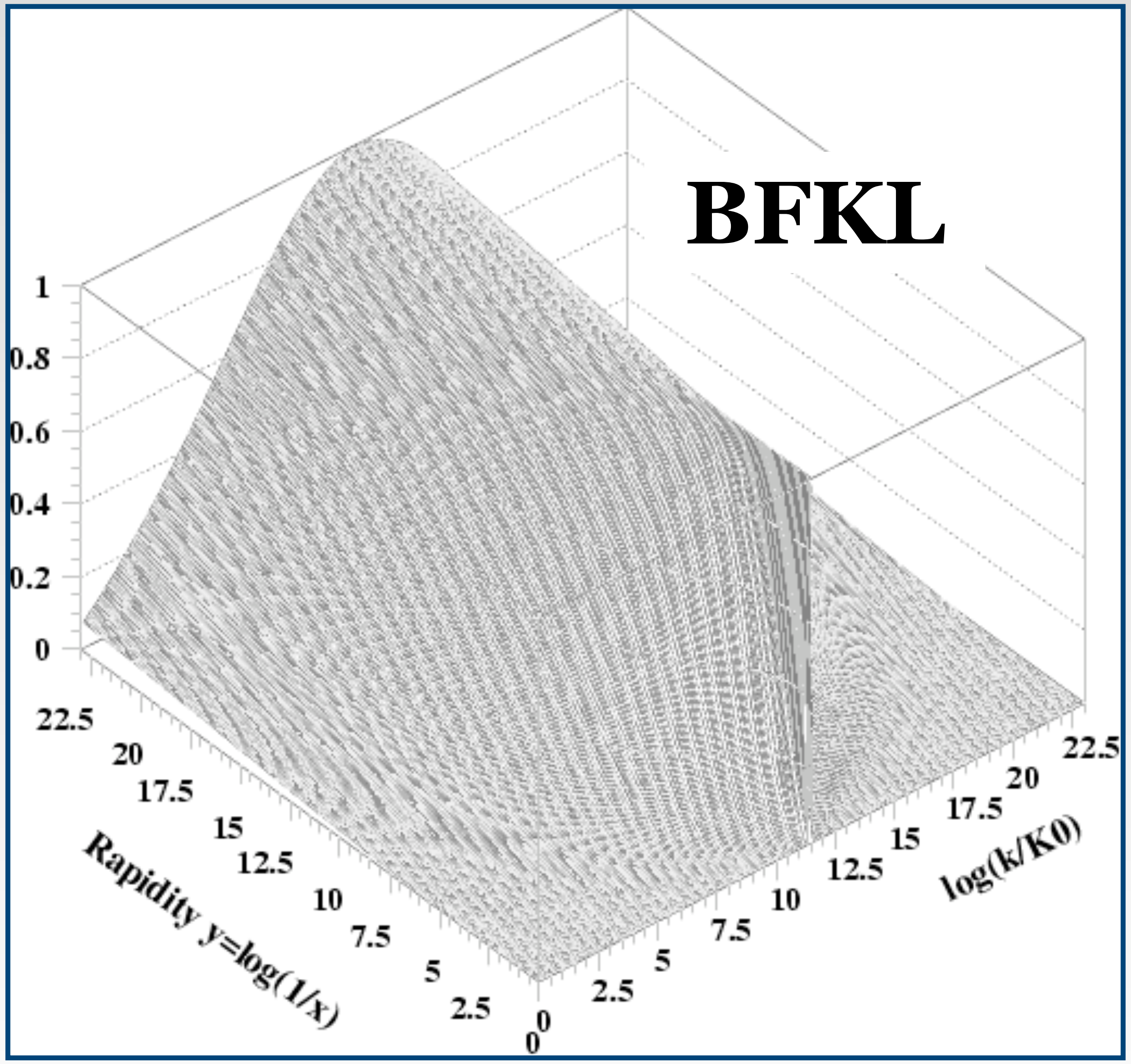
  (Balitsky-Kovchegov equation)

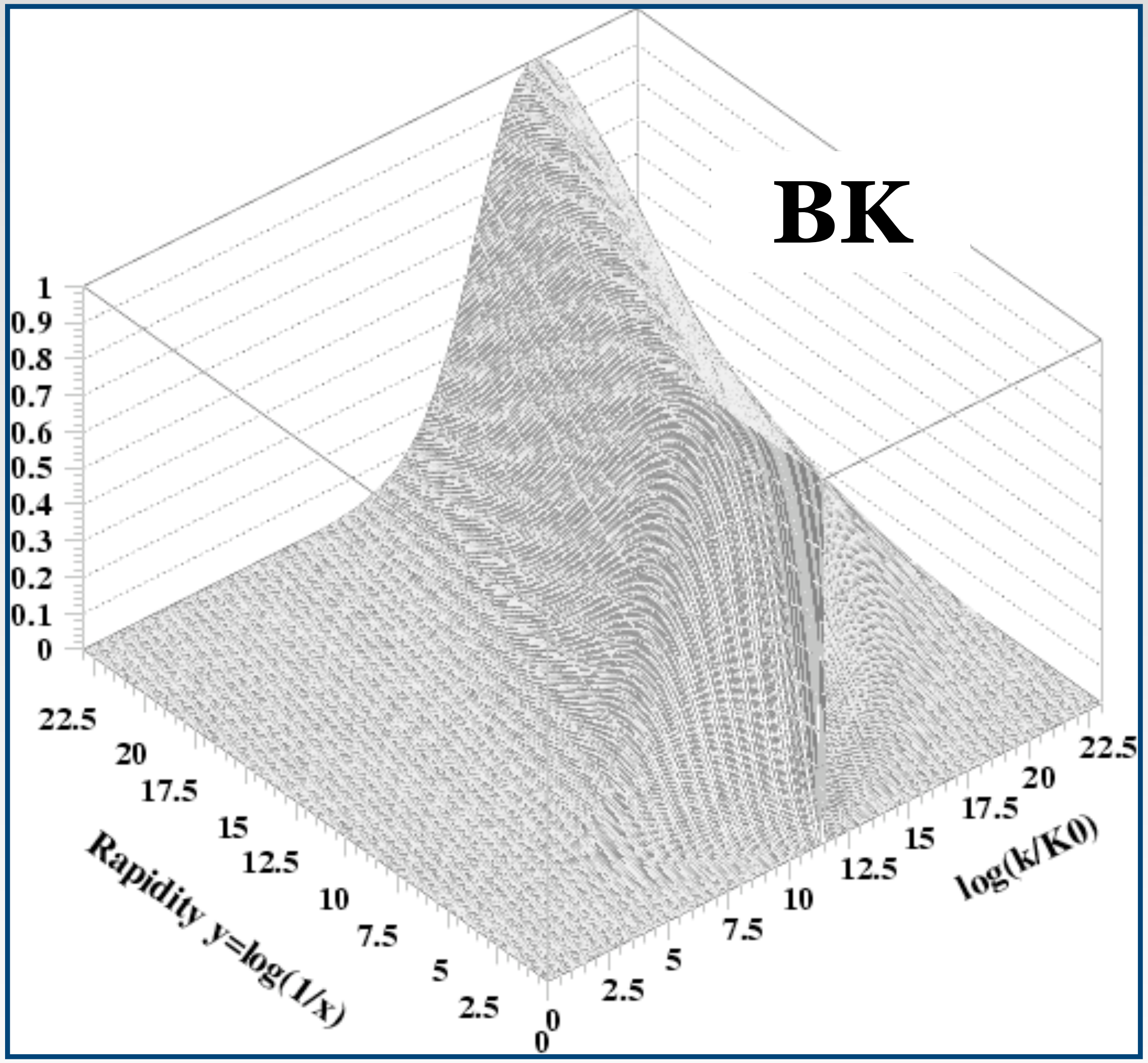
- The r.h.s. vanishes when T reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $T=0$ and $T=1$ are fixed points of this equation

$$T=\epsilon \ : \qquad \text{r.h.s.} > 0 \qquad \Rightarrow \quad T=0 \ \text{ is unstable}$$
$$T=1-\epsilon \ : \qquad \text{r.h.s.} > 0 \qquad \Rightarrow \quad T=1 \ \text{ is stable}$$

Solving the BK equation



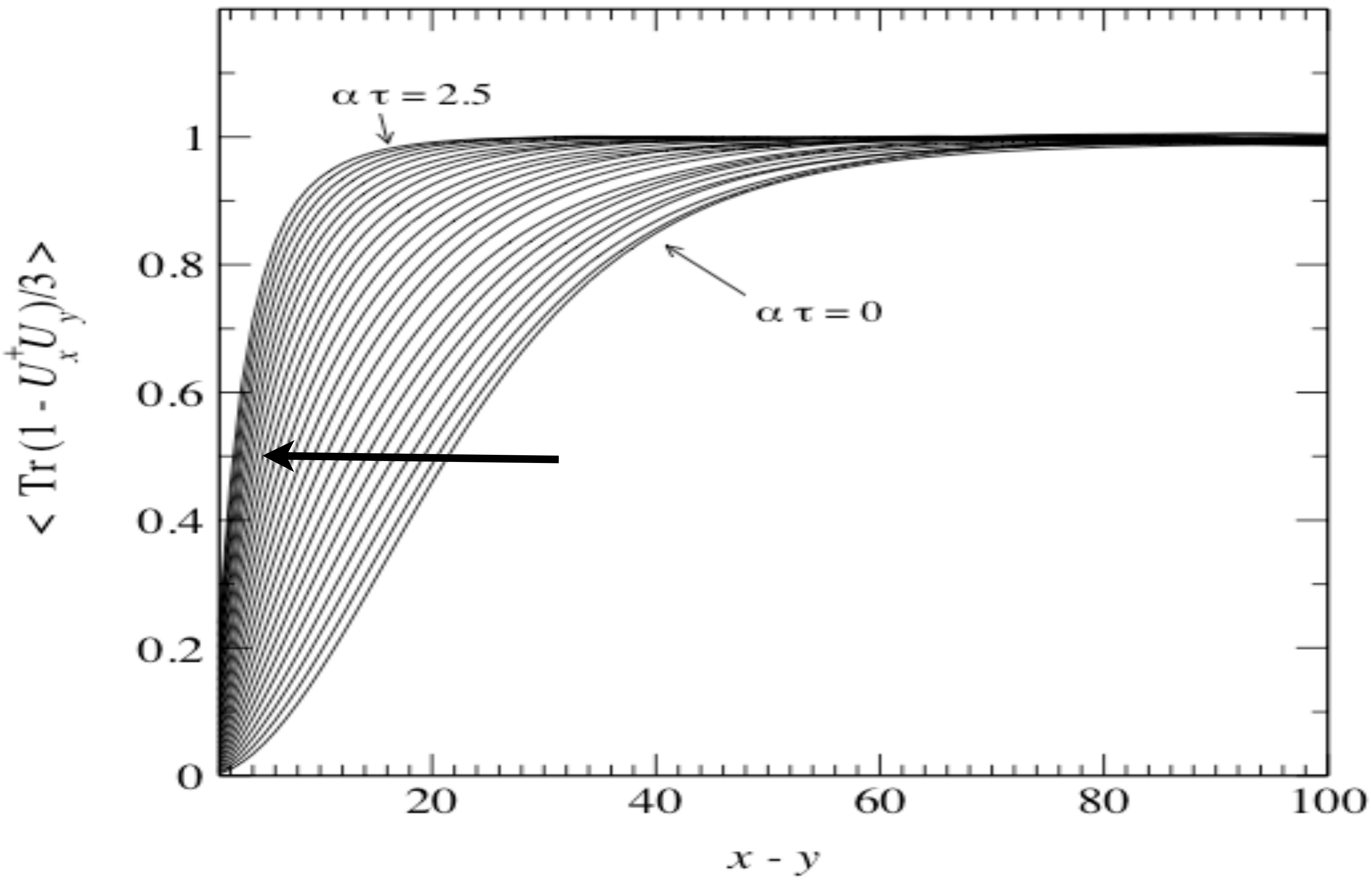


diffusion problem of BFKL is cured by non-linearities

Solving the BK equation

the 2-point function $T(x_t, y_t) = 1/N_c \, \mathrm{Tr}\,[1 - U^{+}(x_t)\, U(y_t)]$

(probability for scattering of a quark-anti-quark dipole on a target)



define $Q_s = 1/r_t$

when $T(r_t) = 1/2$

it grows with energy

color transparency

$$T \sim r_t^2 \, xG(x, \frac{1}{r_t})$$

non-linearities unitarize the scattering probability

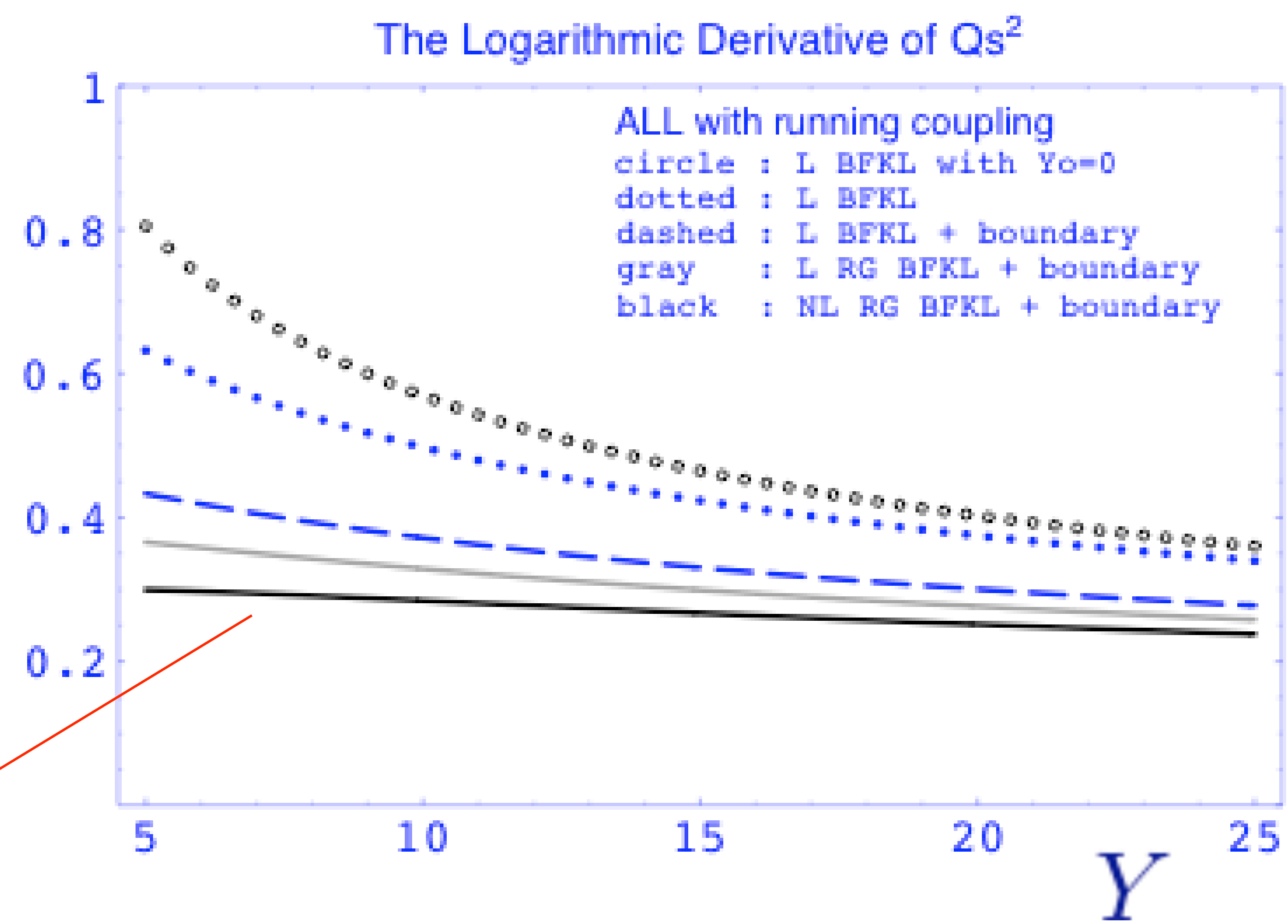
How does Q_s behave as function of Y?

Fixed coupling LO BFKL: $Q_s^2 = Q_0^2 e^{c\,\bar{\alpha}_s Y}$

LO BFKL+ running coupling: $Q_s^2 = \Lambda_{QCD}^2 e^{\sqrt{2b_0 c(Y+Y_0)}}$

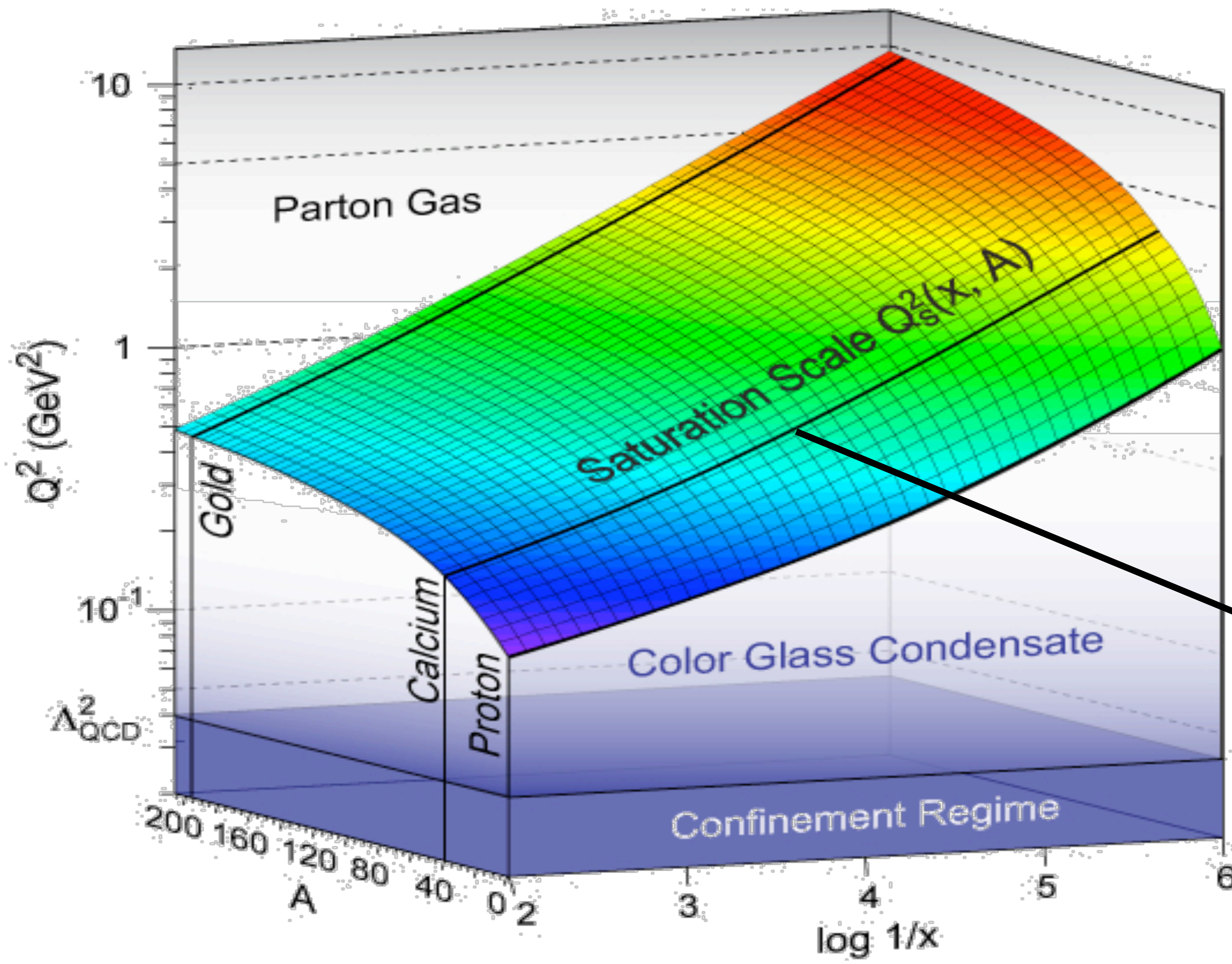
Re-summed NLO BFKL + CGC:

$$\lambda \equiv \frac{d \ln Q_s^2}{dY}$$



Very close to HERA result!

The saturation scale



$\times \frac{9}{4}$ **for gluon**

$\alpha_s(Q_s^2) << 1$

Road map of the strong interactions