QCD for LHC Physics - 3

M. E. Peskin
Sonora Autumn School
October 2008

In the previous lecture, we discussed the importance of obtaining QCD predictions for multi-parton processes.   In principle, this is simple to do -- just compute the Feynman diagrams. In practice, you know that these computations are not simple and can often be long and complicated.

Methods have been developed that dramatically simplify the computation of QCD amplitudes. These methods also reveal unexpected properties of the amplitudes themselves that provide further simplification.

I will describe some of these methods in this lecture. I will concentrate on the computation of tree amplitudes. Similar techniques can be applied to the simplification of loop amplitude computations.

A key feature of these methods is the analysis of scattering amplitudes between particles of definite helicity.

We have seen that this point of view already simplifies the analysis of the basic QCD scattering processes. The simplifications are even greater for more complex processes.

Many of these developments are reviewed in articles of Mangano and Parke, Physics Reports, 1991, and Dixon, 1995 TASI lectures. I will describe also some new methods that so beyond the work described in these papers.

To begin, note that computations with massless particles can be dramatically simplified by the use of spinors of lightlike momenta

$$1\rangle = u_R(1) \quad 1] = u_L(1) \quad \langle 1 = \overline{u}_L(1) \quad [1 = \overline{u}_R(1)$$

These objects are related to more familiar objects by

$$1\rangle[1 = \frac{1}{2}(1+\gamma^5)\not{1}$$

The spinor products are square roots of Lorentz vector products:

$$\langle 12\rangle = \overline{u}_L(1)u_R(2) \qquad [12] = \overline{u}_R(1)u_L(2)$$

$$|\langle 12\rangle|^2 = |[12]|^2 = 2k_1\cdot k_2$$

The spinor products are antisymmetric. They obey the following useful identities:

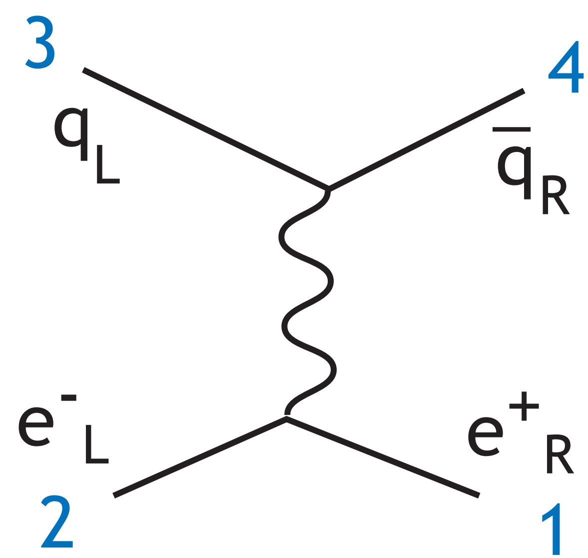
$$\langle 1\gamma^\mu 2]\langle 3\gamma_\mu 4] = 2\,\langle 13\rangle[42] \qquad \text{Fierz}$$

$$\langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 42\rangle + \langle 14\rangle\langle 23\rangle = 0 \quad \text{Schouten}$$

It is simplest to label the helicities as if all particles were outgoing. An incoming L corresponds to an outgoing R. This makes crossing symmetry automatic.

Here is a very simple example: $e^-_L e^+_R \to q_L \overline{q}_R$

$$
\begin{aligned}
iM &= (-ie)^2 \langle 1\gamma^\mu 2] \frac{-i}{s_{34}} \langle 3\gamma_\mu 4] \\
&= 2ie^2 \frac{\langle 13\rangle[42]}{\langle 34\rangle[43]} \times \frac{\langle 31\rangle}{\langle 31\rangle} \\
&= -2ie^2 \frac{(\langle 13\rangle)^2}{\langle 34\rangle\langle 12\rangle}
\end{aligned}
$$



Then

$$
|M|^2 = 4e^4 \frac{u^2}{s^2} = e^4(1+\cos\theta)^2
$$

which is correct !

Photon and gluon polarization vectors for momentum k are conveniently written in terms of spinor products involving k and a second lightlike "reference" vector r:

$$\epsilon_+^\mu(k) = \frac{1}{\sqrt{2}} \frac{\langle r\gamma^\mu k]}{\langle rk\rangle} \qquad \epsilon_-^\mu(k) = -\frac{1}{\sqrt{2}} \frac{[r\gamma^\mu k\rangle}{[rk]}$$

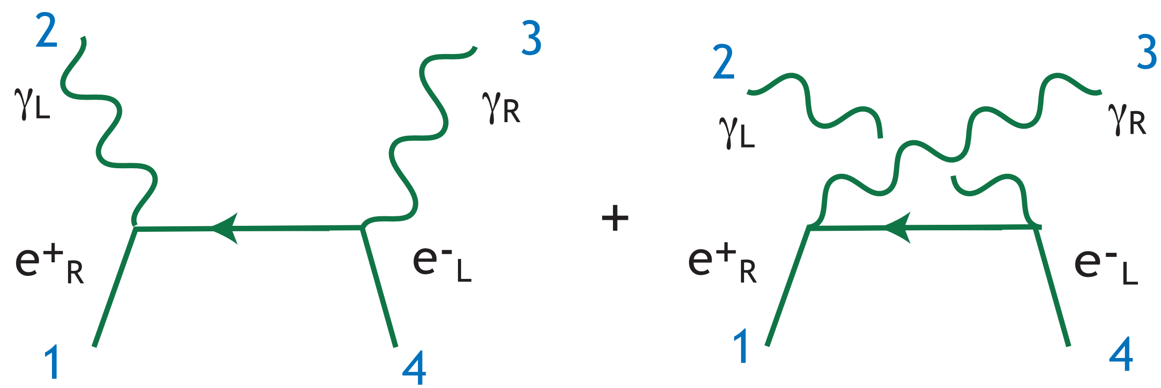
The logic of this choice is:

if k = 1 is parallel to z, r = 2 is parallel to -z:

$$\langle 2\gamma^\mu 1] = \overline{u}_L(2)\gamma^\mu u_L(1) = \sqrt{s_{12}} \cdot (0, 1, -i, 0)$$

a change in r is like a change of gauge and has no physical effect. We can choose r for maximum convenience, independently in each helicity amplitude.

Another example:



choose r = 1 for 3
r = 4 for 2

second diagram:

$$\langle 1\gamma^\mu \cdots \frac{\langle 1\gamma_\mu 3]}{\langle 13\rangle} \cdots = \langle 11\rangle \cdots = 0$$

first diagram:

$$(-ie^2)\langle 1\gamma^\mu \frac{(1+2)}{s_{12}}\gamma^\nu 4] \cdot \frac{1}{\sqrt{2}}\frac{[4\gamma_\mu 2\rangle}{[42]} \cdot \frac{-1}{\sqrt{2}}\frac{\langle 1\gamma_\nu 3]}{\langle 13\rangle}$$

$$= 2ie^2 \frac{\langle 12\rangle[4(1+2)1\rangle[34]}{[42]\langle 12\rangle[21]\langle 13\rangle}$$

$$= 2ie^2 \frac{\langle 21\rangle[34]}{[21]\langle 13\rangle}$$

$$= 2ie^2 \left(\frac{t}{u}\right)^{1/2}$$

which is correct!

We can incorporate massive vector bosons such as W by decaying them to massless fermions (Kleiss and Stirling). Schematically, replace

$$\epsilon^\mu(Q) \to \overline{u}_L(1)\gamma^\mu u_L(2) = \langle 1\gamma^\mu 2]$$

To correctly normalize the result, consider

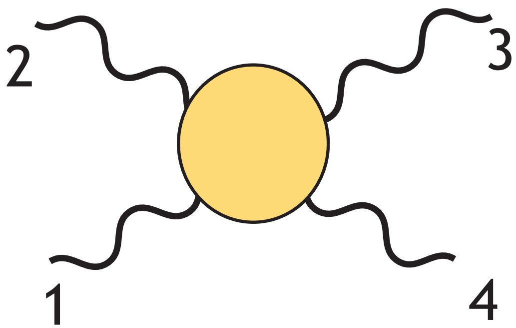
$$\begin{aligned} \int \frac{d\Omega}{4\pi}\langle 1\gamma^\mu 2]\langle 2\gamma^\nu 1] &= A(\eta^{\mu\nu} - Q^\mu Q^\nu/Q^2) \\ \text{contract:} \quad \int \frac{d\Omega}{4\pi}\; 2\langle 12\rangle[12] &= A \cdot 3 \end{aligned}$$

This shows that

$$\frac{3}{2m_W^2}\int \frac{d\Omega}{4\pi}\langle 1\gamma^\mu 2]\langle 2\gamma^\nu 1] = \sum_i \epsilon^\mu(Q)\epsilon^{*\nu}(Q)$$

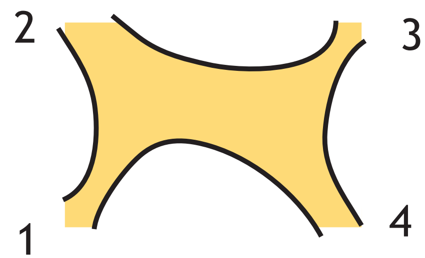
As a bonus, the generated final state fermions correctly represent the polarization of the W's. A similar trick works for top quarks.

Now think about QCD. It is very convenient to write QCD amplitudes in terms of color structures



$$= tr[T_1 T_2 T_3 T_4] A(1,2,3,4) + \cdots$$

=



+ 5 other color structures

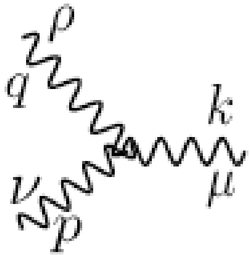
To do this, write

$$t^a = \frac{1}{\sqrt{2}} T^a \quad i f^{abc} = \frac{-i}{\sqrt{2}} tr[T^a T^b T^c - T^a T^c T^b]$$

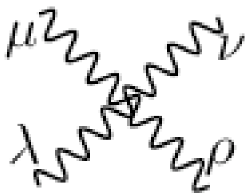
To leading order in N_C , the color-ordered amplitudes A do not interfere. Square them and multiply by the power of N_C indicated by the number of color loops.

Color-ordered amplitudes can be computed with the color-ordered Feynman rules:

$$= \frac{ig}{\sqrt{2}}\gamma^{\mu}$$

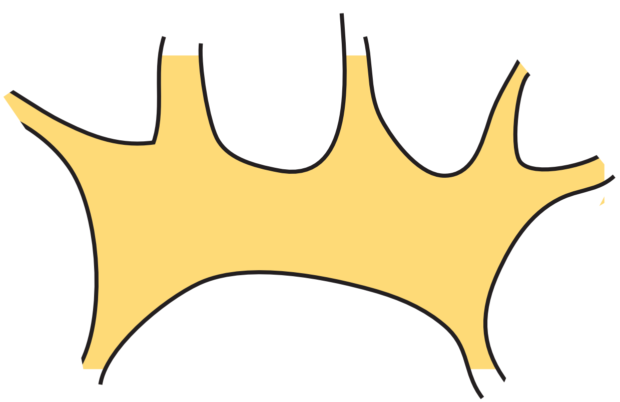


$$= \frac{ig}{\sqrt{2}}\left(\eta^{\mu\nu}(k-p)^{\rho} + \eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu}\right)$$



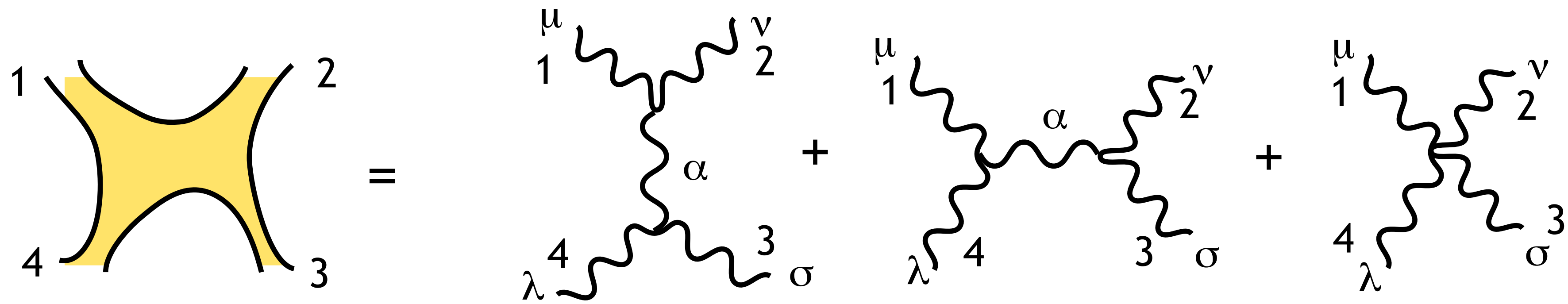
$$= \frac{ig^{2}}{2}\left(2\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}\right)$$

It can be shown that given color-ordered amplitude has only the singularities from collinear singularities in its color-ordering:



$$\sim \frac{1}{(2k_1\cdot k_2)(2k_2\cdot k_3)\cdots(2k_n\cdot k_1)}$$

Illustrate this for the 4-gluon amplitude.



$$
\begin{aligned}
= (\frac{ig}{\sqrt{2}})^2 \quad \Big[\quad & \frac{-i}{s_{12}}[g^{\mu\nu}(1-2)^\alpha + 2g^{\nu\alpha}2^\mu - 2g^{\alpha\mu}1^\nu][g^{\sigma\lambda}(3-4)^\alpha + 2g^{\lambda\alpha}4^\sigma - 2g^{\alpha\sigma}3^\lambda] \\
+ \quad & \frac{-i}{s_{41}}[g^{\lambda\mu}(4-1)^\alpha + 2g^{\mu\alpha}1^\mu - 2g^{\alpha\lambda}4^\nu][g^{\nu\sigma}(2-3)^\alpha + 2g^{\sigma\alpha}3^\sigma - 2g^{\alpha\nu}2^\lambda] \\
+ \quad & (-i)[2g^{\mu\sigma}g^{\nu\lambda} - g^{\mu\nu}g^{\lambda\sigma} - g^{\lambda\mu}g^{\nu\sigma}] \quad \Big] \\
& \cdot\ \epsilon_\mu(1)\epsilon_\nu(2)\epsilon_\sigma(3)\epsilon_\lambda(4)
\end{aligned}
$$

Every term contains at least one $g^{\alpha\beta}$. For all + helicities, choose the same reference vector r for all gluons. Then use

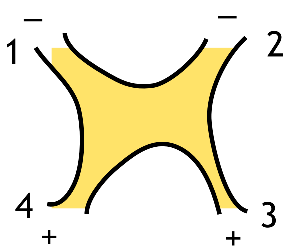
$$
\epsilon_+^\mu(1)\epsilon_{+\mu}(2) = \frac{1}{2}\frac{\langle r\gamma^\mu 1]\langle r\gamma_\mu 2]}{\langle r1\rangle\langle r2\rangle} = \frac{\langle rr\rangle\langle 12\rangle}{\langle r1\rangle\langle r2\rangle} = 0
$$

The whole amplitude is zero!

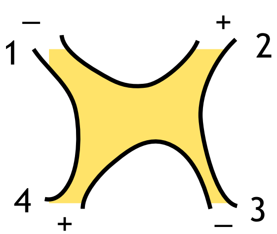
This argument works for any n-gluon amplitude with all + helicities.

Working a little harder, one can show that all amplitudes with one - helicity and all others + also vanish.

There are two types of nonvanishing amplitudes:



$$= -ig^2 \frac{\langle 12\rangle^1 [34]^2}{s_{12}s_{14}} = ig^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$



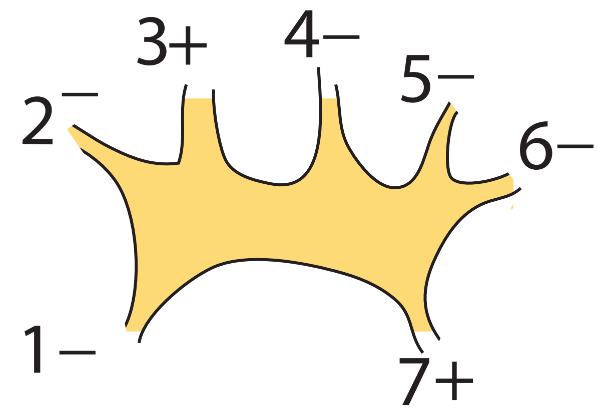
$$= ig^2 \frac{\langle 13\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

There is a strange property here that we found also in the $e^-_L e^+_R \to q_L \bar{q}_R$ example. The final answer has only angle brackets, with no square brackets.

Parke and Taylor showed that there is a general property here that applies to tree amplitudes with arbitrarily many gluons:

Notate:

$$i\mathcal{M}(1^-,2^-,3^+,4^-,5^-,6^-,7^+) =$$



Then:

All amplitudes with all + or only one - vanish. Similarly, all amplitudes with all - or only one + vanish.

Amplitudes with two - and all the rest + have the following simple form:

$$i\mathcal{M}(1^+\ldots i^-\ldots j^-\ldots n^+) = ig^{n-2}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$$

These are called Maximum Helicity Violating (MHV) amplitudes.

This result applies not only to multi-gluon amplitudes but also to other amplitudes of interest. Here are two more MHV formulae:

$$i\mathcal{M}(q_L g_2^+ g_3^+ \cdots g_i^- \cdots g_{n-1}^+ \overline{q}_R) = i g^{n-2} \frac{\langle 1i \rangle^3 \langle ni \rangle}{\langle 12 \rangle \langle 23 \rangle \cdots \langle n1 \rangle}$$

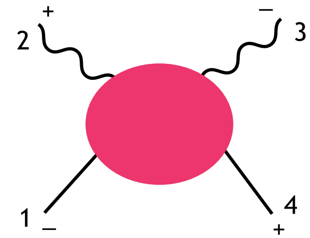
$$i\mathcal{M}(e_L^- e_R^+ q_L g_4^+ g_5^+ \cdots g_{n-1}^+ \overline{q}_R) = -i g_w^2 g^{n-4} \frac{\langle 13 \rangle^2}{\langle 12 \rangle \, \langle 34 \rangle \, \langle 45 \rangle \cdots \langle n-1\, n \rangle}$$

The second formula crosses into an MHV formula for

$$u\overline{d} \to W^+ + ng \to \ell^+ \nu + ng$$

and we can use this to compute heavy particle production at LHC.

I will sketch a derivation of these formulae, but, first, consider some applications. For example, derive the cross section for $u\overline{u} \to gg$.



$$i\mathcal{M}(u^-(1)g^+(2)g^-(3)\overline{u}^+(4))$$
$$= ig^2\left[T^aT^b\frac{\langle 13\rangle^3\langle 43\rangle}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle} + T^bT^a\frac{\langle 13\rangle^3\langle 43\rangle}{\langle 13\rangle\langle 32\rangle\langle 24\rangle\langle 41\rangle}\right]$$

Square the factor in brackets, and use $u^2 + t^2 = s^2 - 2ut$ to find:

$$[\frac{u^3t}{t^2s^2} + \frac{u^3t}{u^2s^2} - \frac{1}{4}\frac{u^3t}{uts^2}] = [\frac{u}{t} - (2+\frac{1}{4})\frac{u^2}{s^2}]$$

Add the square of the other nonzero amplitude; this gives finally:

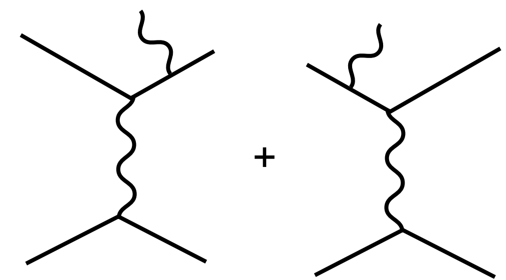
$$[\frac{u}{t} + \frac{t}{u} - \frac{9}{4}\frac{u^2+t^2}{s^2}]$$

Next, derive the cross section for $e^+e^- \to q\bar{q}g$

$$i\mathcal{M}(e^-(1)\overline{e}^+(2)q^-(3)g^+(4)\overline{q}^+(5)) = e^2 g \frac{\langle 13\rangle^2}{\langle 12\rangle\langle 34\rangle\langle 45\rangle}$$

The square of this gives:

$$\frac{s_{13}^2}{s \cdot s(1-x_{\overline{q}}) \cdot s(1-x_q)}$$



Average over the orientation of the 12 axis, and add the amplitude of (g-(4)) to find

$$\frac{x_q^2 + x_{\overline{q}}^2}{(1-x_{\overline{q}})(1-x_q)}$$

There are nonzero 3-point MHV amplitudes.

$$i\mathcal{M}(1^-2^-3^+) = ig\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \qquad i\mathcal{M}(1^+2^+3^-) = -ig\frac{[12]^4}{[12][23][31]}$$

These are on-shell 3-gluon amplitudes. It seems odd that such amplitudes could make sense because

$$3^2 = s_{12} = \langle 12\rangle[21] = 0$$

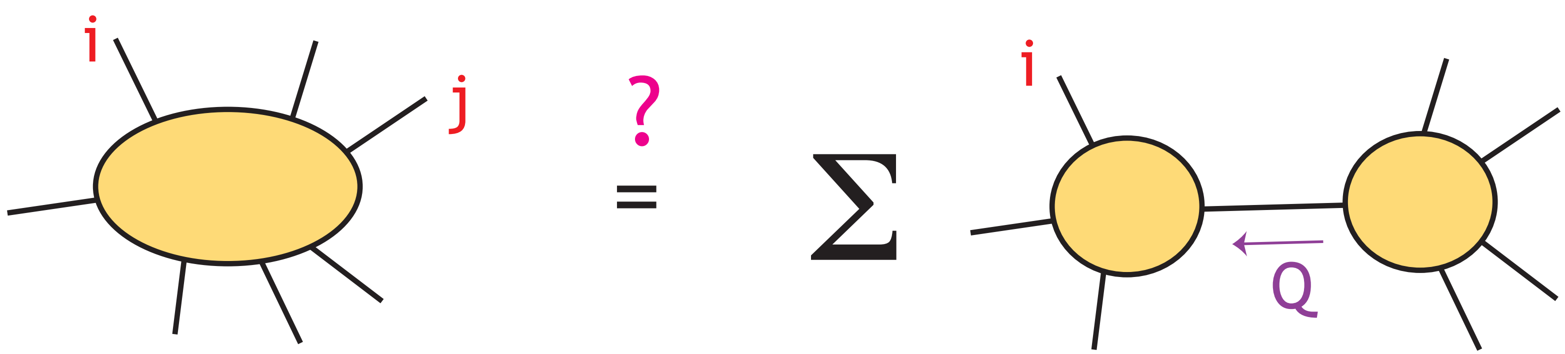
However, if we allow external momenta to be complex, this condition implies only
$$\langle 12\rangle \text{ or } [21] = 0$$

so there are complex momenta for which these are proper on-shell amplitudes.

I will now show that these 3-point amplitudes are building blocks that can be used to construct the most general on-shell n-gluon amplitudes.

This is part of a general question: How do we use the MHV amplitudes to derive n-gluon amplitudes for more general helicity states ? I will now describe a beautifully simple method, discovered by Britto, Cachazo, and Feng.

It would be wonderful to build up non-MHV amplitudes from the simple MHV expressions. But it is not so obvious how to do this. MHV amplitudes are on-shell expressions, but if we cut the more complex amplitudes, we will have to evaluate them off-shell.



Or do we ? BCF suggested that we pick legs i and j and shift

$$i] \to i] + z\ j] \qquad j\rangle \to j\rangle - z\ i\rangle$$

for z a complex variable. Then the shifted Q can be on-shell.

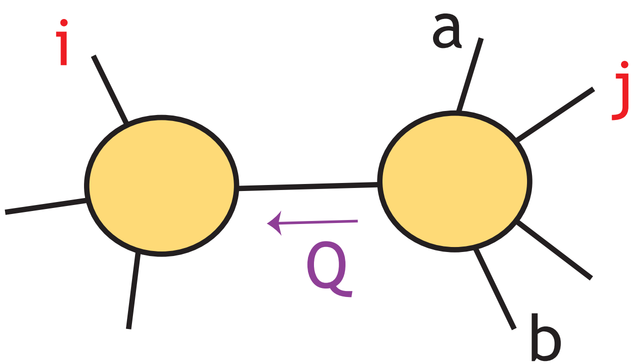
Now consider

$$\oint \frac{dz}{2\pi i}\ \frac{i\mathcal{M}(z)}{z} = i\mathcal{M}(z=0) + (\text{other poles}) = (\text{contour at } \infty)$$

The first term is the amplitude that we wish to evaluate. The contour at ∞ vanishes if we choose i and j correctly (e.g i a − gluon and j a + gluon). The additional poles result when a momentum on an intermediate line satisfies

$$Q(z)^2 = 0$$

Looking again at the diagram,



$$Q^\mu(z)\gamma_\mu = \sum_{k=a}^{b} k\rangle[k \ - \ z\ i\rangle[j$$

and so

$$z_* = \frac{s_{a\ldots b}}{\langle i(\sum k\rangle[k)j]}$$

Tidying up the formula, one finds the following relation:

$$i\mathcal{M}(1\cdots n) \quad = \quad \sum_{splits} i\mathcal{M}(b+1 \cdots \hat{i} \cdots a-1 \; -\hat{Q}) \cdot \frac{1}{s_{a\cdots b}} \cdot i\mathcal{M}(a \cdots \hat{j} \cdots b\,\hat{Q})$$

called the Britto-Cachazo-Feng (BCF) recursion formula.

Momenta with hats have the shift with z_* . The hatted momenta are complex but satisfy $\hat{Q}^2 = 0$, so the amplitudes on the right-hand side are to be evaluated on shell!

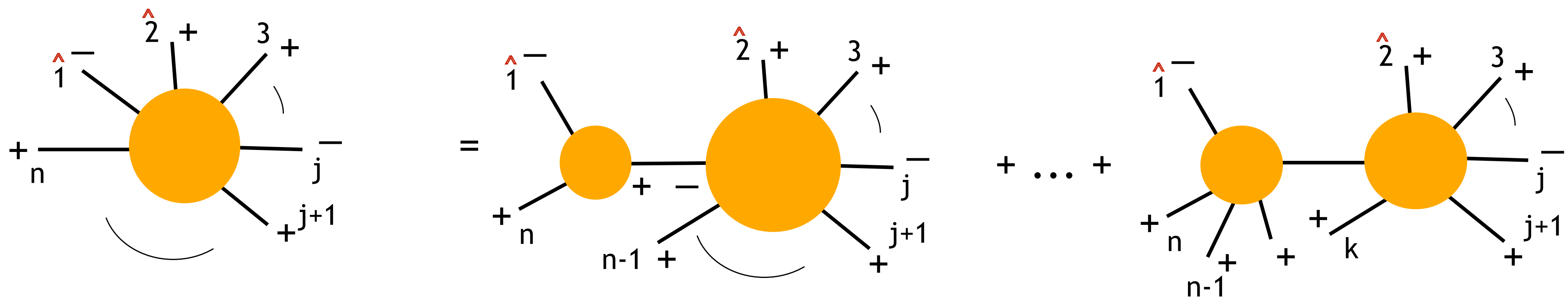
This allows the n-point amplitudes to be recursively evaluated in terms of amplitudes with fewer legs. We can stop when we reach MHV. At 5 points all amplitudes are MHV or anti-MHV.

As an illustration, derive the n-gluon MHV amplitude from the (n-1)-gluon MHV amplitude: Let 1, j be the - helicities. Apply the shift:

$$\hat{1}] = 1] + z\ 2] \quad \hat{2}\rangle = 2\rangle - z\ 1\rangle$$

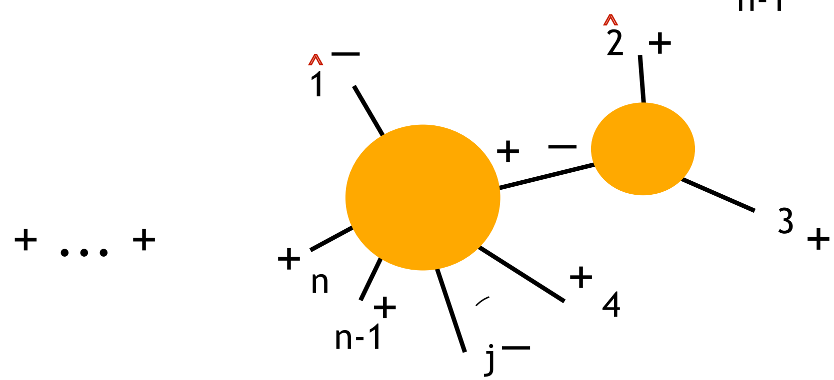
The BCF recursion is

Risager



The first diagram on the right is zero, because

$$\hat{1}^2 = 0 \rightarrow [n\hat{Q}] = 0$$



So only the last diagram is nonzero.

The value of this diagram is

$$ig^{n-3}\frac{\langle 1j\rangle^4(-1)}{\langle 1\hat{Q}\rangle\langle\hat{Q}4\rangle\langle 45\rangle\cdots\langle n1\rangle}\frac{i}{\langle 23\rangle[32]}(-ig)\frac{[23]^4}{[\hat{Q}2][23][3\hat{Q}]}$$

with

$$\hat{Q} = -2\rangle[2 - 3\rangle[3 + z\ 1\rangle[2$$

This implies

$$\begin{aligned}\langle 1\hat{Q}\rangle[\hat{Q}3] &= -\langle 12\rangle[23] \\ \langle 4\hat{Q}\rangle[\hat{Q}2] &= -\langle 43\rangle[32]\end{aligned}$$

The factors of $[23]$ all cancel, and we find

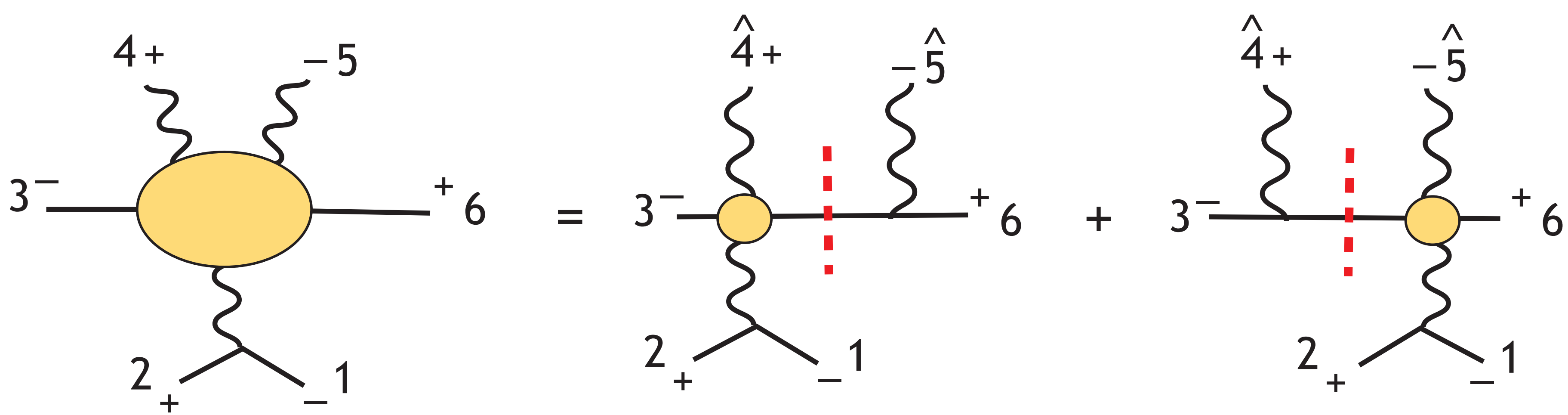
$$ig^{n-2}\frac{\langle 1j\rangle^4}{\langle 12\rangle\langle 34\rangle\langle 45\rangle\cdots\langle n1\rangle\cdot\langle 23\rangle}$$

By induction, the Parke-Taylor MHV result is true for all n.

I would like to illustrate the use of the BCF recursion formula by computing a non-MHV contribution to $e^+e^- \to \bar{q}q + 2 \text{ gluons}$. This crosses into a contribution to W + 2 jets, which we would need to compute the mass distribution of the hadronic system recoiling against a W or Z.

Using the shift $\hat{5}] = 5] + z\, 4] \quad \hat{4}\rangle = 4\rangle - z\, 5\rangle$

we find



I will evaluate the first of these BCF cuts.

$$-ig_w^2 g \frac{\langle 13\rangle^2(-1)}{\langle 12\rangle\langle 3\hat{4}\rangle\langle \hat{4}\hat{Q}\rangle} \;\; \frac{i}{s_{56}} \;\; ig \frac{\langle \hat{Q}5\rangle^3\langle 65\rangle}{\langle \hat{Q}5\rangle\langle 56\rangle\langle 6\hat{Q}\rangle}$$

$$= ig_w^2 g^2 \frac{\langle 13\rangle^2\langle 5\hat{Q}\rangle^2}{\langle 12\rangle\langle 3\hat{4}\rangle\langle \hat{4}\hat{Q}\rangle\langle 6\hat{Q}\rangle}$$

where $\quad \hat{Q} = -5\rangle[5 - 6\rangle[6 - z\, 5\rangle[4$

$$\hat{Q}^2 = 0 \text{ for } z = -\frac{s_{56}}{\langle 5(5+6)4]} = -\frac{[65]}{[64]}$$

Multiply the formula top and bottom by $[\hat{Q}4]^2$ and work out the pieces:

$$\begin{aligned} \langle 5\hat{Q}\rangle[\hat{Q}4] &= -\langle 56\rangle[64] \\ \langle 6\hat{Q}\rangle[\hat{Q}4] &= -\langle 65\rangle[54] \\ \langle \hat{4}\hat{Q}\rangle[\hat{Q}4] &= -s_{456} \\ \langle 3\hat{4}\rangle &= \langle 3(4+5)6]/[46] \end{aligned}$$

in all

$$-ig_w^2 g^2 \frac{\langle 13\rangle^2 [46]^3}{\langle 12\rangle [45][56] s_{456} \langle 3(4+5)6]}$$

combining with the other BCF cut, the complete amplitude is

$$-ig_w^2 g^2 \quad \left\{ \quad \frac{\langle 13\rangle^2 [46]^3}{\langle 12\rangle [45][56] s_{456} \langle 3(4+5)6]} \right.$$

$$\left. + \quad \frac{[26]^2 \langle 35\rangle^3}{[12]\langle 34\rangle \langle 45\rangle s_{345} \langle 3(4+5)6]} \right\}$$

In practice, there is no need to write such expressions explicitly. They can be generated by automating the BCF recursion, using the appropriate complex momenta directly.

These and other new tools for QCD perturbation theory will make it easier to compute multi-parton cross sections of interest for LHC. I hope you will find some interesting and useful applications of these techniques.