

String theory basics for non-string-theory people

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Plan for the mini-course

- Motivation
- First part
 - The relativistic point particle
 - The relativistic string 
 - The action
 - Closed string quantization
 - Open string quantization
- Second part
 - D-branes
- Third part
 - The AdS/CFT correspondence

Motivation

Small things, big problems

One of the **greatest problems** of theoretical physics is the incompatibility of Einstein's General Relativity and the principles of Quantum Mechanics.

→ We are searching for a quantum theory of gravity

The Standard Model of particle physics, despite its great success, can not be last word, there are a lot of open questions, e.g.,:

→ Why so many parameters (more than 20)?

→ Why 26 fields? Why 3 generations?

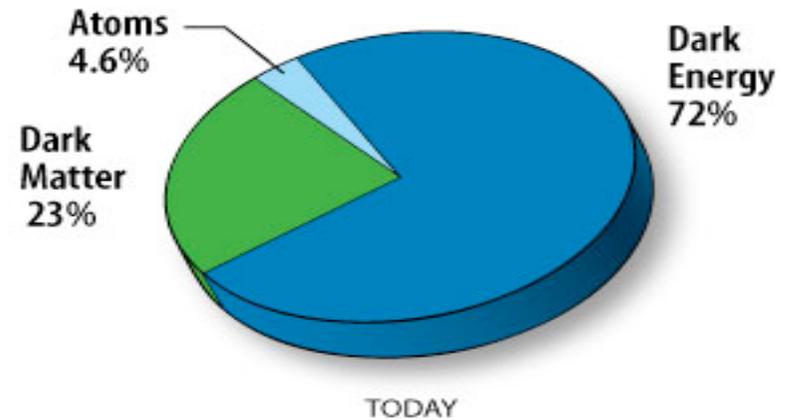
→ Hierarchy problem

→ How do we describe QCD at low energies?

Big things, also big problems

An important question that needs to be answered is what is our universe made of?

- Dark matter (23%)?
- Dark energy (72%)?



None of these questions seems to have a simple answer
Fortunately, a lot of people with great ideas and very different approaches are trying to solve the puzzles...

One of these roads is STRING THEORY

Why string theory?

Pros:

- String theory is a promising candidate (at least for some people) for the long-sought quantum mechanical theory of gravity.
- String theory has the potential to unify the four fundamental forces of nature.
- Interesting new physics (extra dimensions, supersymmetry, more fields, etc)
- A new tool to study certain strongly coupled gauge theories:
The AdS/CFT correspondence

Why string theory?

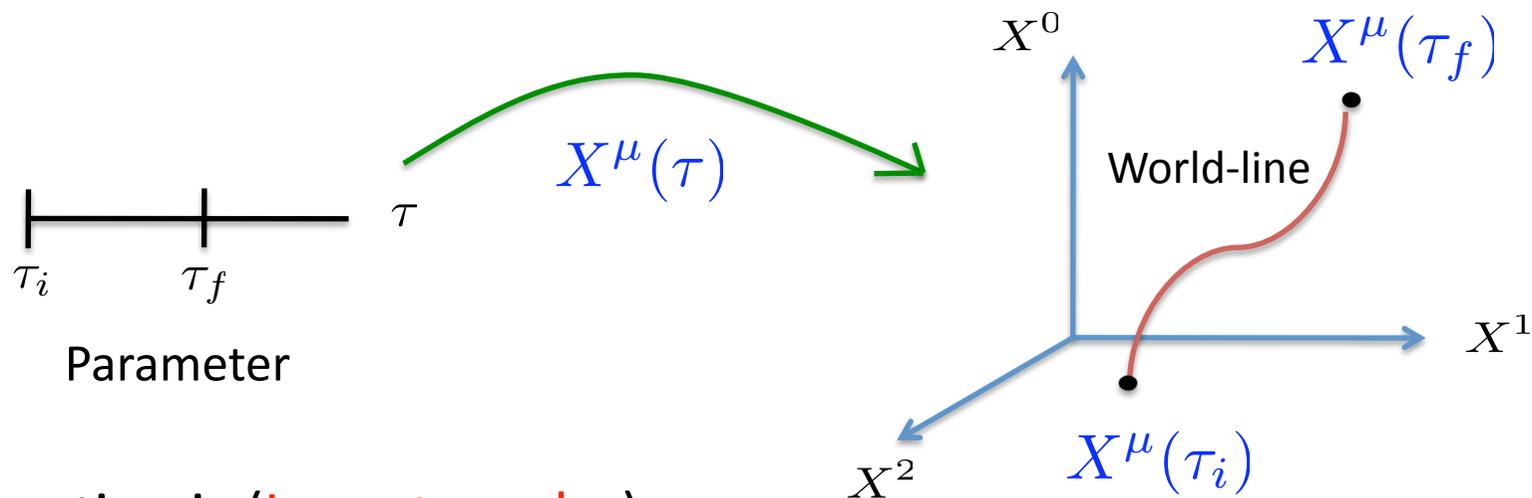
Cons:

- No direct experimental evidence
- It is far from certain that it describes our world
- String theory has not been able to obtain the Standard Model (similar theories)
- The complete theory still unknown. Lack of a non-perturbative definition
- 10 dimensions?

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The relativistic point particle

To preserve manifest Lorentz covariance, we use parameterized description $X^\mu(\tau)$:



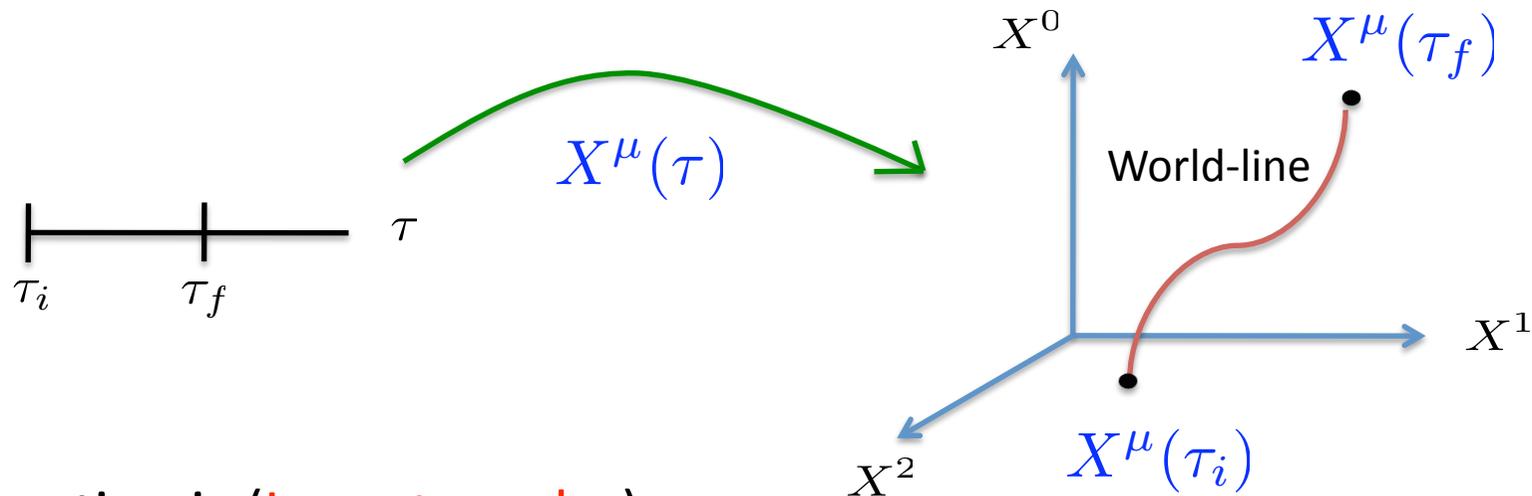
The action is (**Lorentz scalar**):

$$S = -m \times (\text{proper length})$$

$$S[X] = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-g_{\tau\tau}}$$

The relativistic point particle

To preserve manifest Lorentz covariance, we use parameterized description $X^\mu(\tau)$:



The action is (**Lorentz scalar**):

$$S[X] = -m \int_{\tau_i}^{\tau_f} d\tau \sqrt{-G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu}$$

where $G_{\mu\nu}$ = spacetime metric

and $g_{\tau\tau} = G_{\mu\nu} \partial_\tau X^\mu \partial_\tau X^\nu$ = induced metric on worldline

The symmetries of this action:

- **Spacetime reparametrization invariance** (if $G_{\mu\nu} = \eta_{\mu\nu}$ then Poincaré invariance)
- **Worldline reparametrization invariance** $\tau \rightarrow \tau'(\tau)$

As usual, we define:

$$P_\mu = \frac{\partial L}{\partial(\partial_\tau X^\mu)} = \frac{-m\dot{X}^\mu}{\sqrt{-\dot{X}^2}}$$

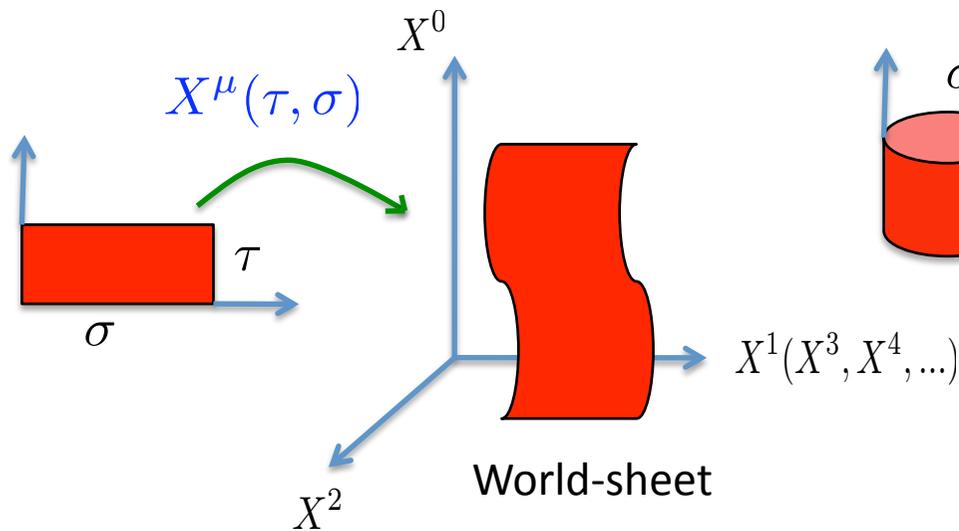
and P_μ satisfies the condition: $P_\mu P^\mu + m^2 = 0$ (first class const.)

➔ D-1 degrees of freedom.

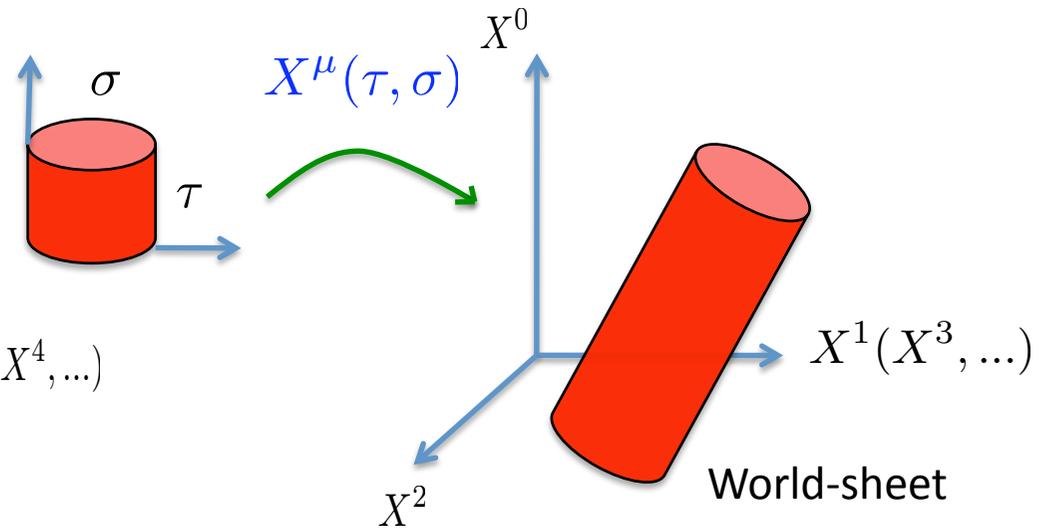
Can we generalize this to a 1-dimensional object??

The relativistic bosonic string

Open strings



Closed strings



The world-sheet is described by the embedding functions: $X^\mu(\tau, \sigma)$

And in complete analogy with the relativistic point particle:

$$S[X] = -T \times (\text{proper area}) \quad (T \equiv \text{tension})$$

$$S_{NG}[X] = -T \int d\tau d\sigma \sqrt{-\det g_{ab}}$$

$$S_{NG}[X] = -T \int d\tau d\sigma \sqrt{-\det G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu}$$

where $g_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \equiv$ induced metric on the worldsheet

Explicitly,

$$-\det g_{ab} = (\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \quad (\dot{X} \equiv \partial_\tau X \quad X' \equiv \partial_\sigma X)$$

and

$$T \equiv \frac{\text{energy}}{\text{length}} \equiv \frac{1}{2\pi l_s^2} \quad \text{where } l_s \text{ is the fundamental string length}$$

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where $g_{ab} = G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \equiv$ induced metric on the worldsheet

This is known as **the Nambu-Goto action**

What are the symmetries of this action?

The symmetries of the Nambu-Goto action:

- **Spacetime reparametrization invariance** (if $G_{\mu\nu} = \eta_{\mu\nu}$ then Poincaré invariance)
- **Worldsheet reparametrization invariance** $(\tau, \sigma) \rightarrow (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$

Now, we have:

$$\Pi_{\mu} \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = -T \frac{(\dot{X} \cdot X') X_{\mu} - X'^2 \dot{X}_{\mu}}{\sqrt{(\dot{X} \cdot X')^2 - X'^2 \dot{X}^2}}$$

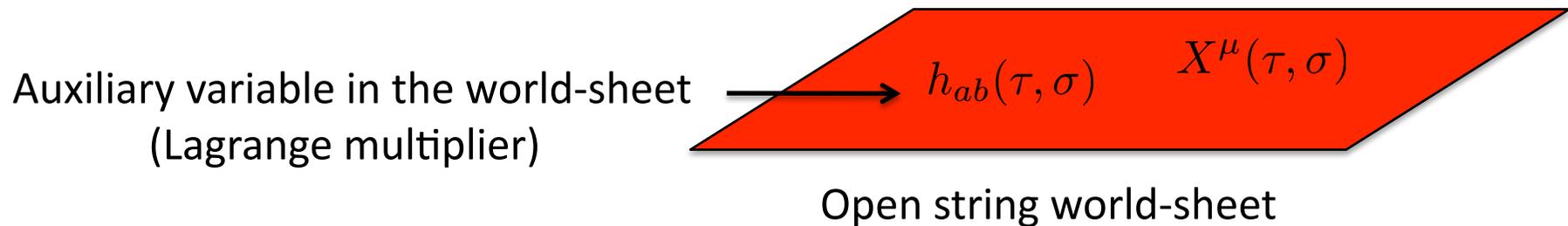
And there are two first class constraints:

$$\left\{ \begin{array}{l} (\Pi_{\mu}) X'^{\mu} = 0 \\ (\Pi_{\mu})^2 + T^2 (X'^{\mu})^2 = 0 \end{array} \right. \longrightarrow \text{(D-2) degrees of freedom}$$

The Nambu-Goto action is non-polynomial, so it is convenient to work with what is known as **the Polyakov action**

$$S_P[X, h_{ab}] = -\frac{T}{2} \int d\tau d\sigma \sqrt{-h} h^{ab} G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \quad (h \equiv \det h_{ab})$$

where $h_{ab} \equiv$ intrinsic metric on the worldsheet , and the other elements are the same as before.



The equation of motion for h_{ab} :

$$\frac{\delta S_P}{\delta h^{ab}} = 0 \quad \longrightarrow \quad T_{ab} = g_{ab} - \frac{1}{2} h_{ab} h^{cd} g_{cd} = 0$$

Solution: $h_{ab}(\sigma, \tau) = \lambda(\sigma, \tau) g_{ab}(\sigma, \tau)$ ($\lambda(\sigma, \tau)$ arbitrary function)

The symmetries of the Polyakov action:

- **Spacetime reparametrization invariance** (if $G_{\mu\nu} = \eta_{\mu\nu}$ then Poincaré invariance)
- **Worldsheet reparametrization invariance** $(\tau, \sigma) \rightarrow (\tau'(\tau, \sigma), \sigma'(\tau, \sigma))$
- **Weyl invariance**: $h'_{ab}(\tau, \sigma) = \Omega(\tau, \sigma)h_{ab}(\tau, \sigma)$

Due to Weyl invariance $T_a^a = 0$, so $h_{ab}(\tau, \sigma)$ instead of three independent components it has only two (first class constraints).

Then, as expected: (D-2) degrees of freedom

Note: for the rest of the talk we will consider $G_{\mu\nu} = \eta_{\mu\nu}$ i.e., Minkowski spacetime, and also $h_{ab} = \eta_{ab}$ (the latter can be found using properties of two-dimensional geometry and worldsheet rep. invariance)

Rewriting the Polyakov action with $G_{\mu\nu} = \eta_{\mu\nu}$ and $h_{ab} = \eta_{ab}$:

$$S_p = -\frac{T}{2} \int d\tau d\sigma (\eta^{ab} \partial_a X \cdot \partial_b X)$$

and the equation of motion is

$$\left(\frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu(\tau, \sigma) = 0 \quad (\text{wave equation!})$$

This is massless Klein-Gordon in (1+1)-dim for D scalar fields

To find a solution we now need to impose the boundary conditions

Boundary conditions

Closed string: $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi) \quad 0 \leq \sigma \leq 2\pi$ (Periodic)

Open string: $0 \leq \sigma \leq \pi$

1- Covariant under Poincaré:

$$\partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0 \quad \forall \tau \quad (\text{Neumann: free endpoints})$$

2- Non covariant under Poincaré

$$\partial_\tau X^i(\tau, 0) = \partial_\tau X^i(\tau, \pi) = 0 \quad \forall \tau \quad i = p + 1, \dots, D - 1$$

$$\longrightarrow X^i(\tau, 0) = X^i(\tau, \pi) = c^i \quad (\text{Dirichlet: fixed endpoints})$$

This last case has very important implications (2nd part).

Closed string quantization

We want to solve the e.o.m. $\partial^2 X^\mu(\tau, \sigma) = 0$ with periodic b.c.

Recall the Fourier mode expansion for a massless scalar field in 1+1 dim

$$\phi(\sigma^0, \sigma^1) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \frac{1}{\sqrt{2E_p}} [a_p e^{ip \cdot \sigma} + a_p^\dagger e^{-ip \cdot \sigma}] \quad \text{with} \quad p \cdot \sigma = -E_p \sigma^0 + p \sigma^1$$

The solution we are looking for is almost a rewriting of the above:

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + i \sqrt{\frac{l_s^2}{2}} \sum_{n \neq 0} \frac{1}{n} (\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)})$$

with:

$$\alpha_n \equiv -i \sqrt{\frac{n}{2\pi}} a_{-n} \qquad \alpha_{-n} \equiv i \sqrt{\frac{n}{2\pi}} a_{-n}^\dagger \qquad \forall n > 0$$

$$\tilde{\alpha}_n \equiv -i \sqrt{\frac{n}{2\pi}} a_n^\dagger \qquad \tilde{\alpha}_{-n} \equiv i \sqrt{\frac{n}{2\pi}} a_n^\dagger \qquad \forall n > 0$$

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The solution we are looking for is almost a rewriting of the above:

$$X^\mu(\tau, \sigma) = \underbrace{x^\mu}_{\text{Position of Center of mass}} + \underbrace{l_s^2 p^\mu \tau}_{\text{Momentum of center of mass}} + i \sqrt{\frac{l_s^2}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\underbrace{\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)}}_{\text{String oscillations}} \right)$$

Right-moving mode

Left-moving mode

Closed string quantization

We want to solve the e.o.m. $\partial^2 X^\mu(\tau, \sigma) = 0$ with periodic b.c.

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The solution we are looking for is almost a rewriting of the above:

$$X^\mu(\tau, \sigma) = x^\mu + l_s^2 p^\mu \tau + i \sqrt{\frac{l_s^2}{2}} \underbrace{\sum_{n \neq 0} \frac{1}{n}}_{\text{Discrete momentum: } p = n \text{ (circle)}} (\alpha_n^\mu e^{-in(\tau+\sigma)} + \tilde{\alpha}_n^\mu e^{-in(\tau-\sigma)})$$

Discrete momentum: $p = n$
(circle)

Doing **canonical quantization**:

$$X^\mu(\tau, \sigma) \text{ and } \Pi^\mu(\tau, \sigma) \quad \longrightarrow \quad \hat{X}^\mu(\tau, \sigma) \text{ and } \hat{\Pi}^\mu(\tau, \sigma)$$

We have: $[\hat{X}^\mu(\tau, \sigma), \hat{\Pi}_\nu(\tau, \sigma')] = i\delta_\nu^\mu \delta(\sigma - \sigma')$

$$\longleftrightarrow \left\{ \begin{array}{l} [\hat{x}^\mu, \hat{p}^\nu] = i\eta_{\mu\nu} \\ [\hat{\alpha}_m^\mu, \hat{\alpha}_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu} \\ [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu} \end{array} \right. \quad \left(\begin{array}{l} \text{identical to:} \\ [\hat{a}_m^\mu, \hat{a}_n^{\dagger\nu}] = 2\pi\delta_{m,n}\eta^{\mu\nu} \end{array} \right)$$

And now we can construct the **Fock space**...

As usual, let's define the vacuum state $|0, 0; k\rangle$ such that:

$$\alpha_n^\mu |0, 0; k\rangle = 0 = \tilde{\alpha}_n^\mu |0, 0; k\rangle \quad \forall n > 0$$

For example, some of the states are:

$$\underbrace{|0, 0; k\rangle}$$

Vacuum: no oscillators

$$\underbrace{(\alpha_n^\mu)^\dagger |0, 0; k\rangle}$$

One left-moving oscillator

$$\underbrace{(\alpha_n^\mu)^\dagger (\alpha_m^\nu)^\dagger |0, 0; k\rangle}$$

Two left-moving oscillators

But, there is a problem: there are states with **NEGATIVE norm**.

For example: $(\alpha_n^0)^\dagger |0, 0; k\rangle \quad \forall n > 0$

$$|(\alpha_n^0)^\dagger |0, 0; k\rangle|^2 = \langle 0, 0; k | \underbrace{\alpha_n^0 (\alpha_n^0)^\dagger} |0, 0; k\rangle = -n(2\pi)^D \delta^D(0)$$

$$[\alpha_n^0, (\alpha_n^0)^\dagger] = n\eta^{00} = -n$$

So what do we do?

Remember: **not all states are physical**

We must impose the constraints

For example, some of the states are:

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Vacuum: no oscillators

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$$[\alpha_n^0, (\alpha_n^0)^\dagger] = n\eta^{00} = -n$$

Meaning that:

$$T_{ab}|\psi\rangle = 0 \quad (\text{analogue to the Gupta-Bleuler method for the Maxwell field})$$

First, recall: $T_{ab} = g_{ab} - \frac{1}{2}h_{ab}h^{cd}g_{cd} = 0$ and $T_a^a = 0$ (Weyl invariance)

Let's define: $\begin{cases} \sigma^+ \equiv \tau + \sigma \\ \sigma^- \equiv \tau - \sigma \end{cases}$ and rewrite T_{ab} :

$$\begin{aligned} T_{++} &= \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X^\mu \partial_+ X_\mu & \partial_a T_{ab} = 0 \\ T_{--} &= \frac{1}{2}(T_{00} - T_{01}) = \partial_- X^\mu \partial_- X_\mu \end{aligned} \quad \longrightarrow \quad \begin{cases} T_{++} \equiv T_{++}(\sigma^+) \\ T_{--} \equiv T_{--}(\sigma^-) \end{cases}$$

and now, we can expand $T_{++}(\sigma^+)$ and $T_{--}(\sigma^-)$ in Fourier modes, i.e.,

$$T_{++}(\sigma^+) = \sum_{n=-\infty}^{n=\infty} L_n e^{-in\sigma^+} \quad \longrightarrow \quad L_n^\dagger = L_{-n}$$

$$T_{--}(\sigma^-) = \sum_{n=-\infty}^{n=\infty} \tilde{L}_n e^{-in\sigma^-} \quad \longrightarrow \quad \tilde{L}_n^\dagger = \tilde{L}_{-n}$$

It is possible to write the **Fourier coefficients** in terms of α_n and α_{-n} (and the right-moving modes):

$$L_m = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \quad \text{and} \quad \tilde{L}_m = \frac{1}{2} \sum_{-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n$$

When dealing with **quantum theories**, one must resolve ordering ambiguities. In our case we just have an ambiguity in L_0 (\tilde{L}_0), one can show that:

$$L_0 - 1 = \frac{1}{2} \alpha_0^2 + \underbrace{\sum_{n>0} \alpha_{-n} \cdot \alpha_n}_{\equiv N} - 1$$

Ordering constant
↖

So, using the constraints translates into:

$$\text{Closed string} \left\{ \begin{array}{l} (L_0 - 1)|\psi\rangle = 0 = (\tilde{L}_0 - 1)|\psi\rangle \\ L_{m>0}|\psi\rangle = 0 = \tilde{L}_{m>0}|\psi\rangle \\ (N - \tilde{N})|\psi\rangle = 0 \end{array} \right.$$

So now we are ready to determine the **physical states**...

From the first condition: $(L_0 - 1)|\psi\rangle = 0 = (\tilde{L}_0 - 1)|\psi\rangle$

and using $M^2 = -p^2$ $(L_0 - 1 = \frac{l_s^2}{4}p^2 + N - 1)$ we find that

$$M^2 = \frac{2}{l_s^2}(N + \tilde{N} - 2)$$

- $N = \tilde{N} = 0 \longrightarrow M^2 = -\frac{4}{l_s^2}$

State: $|0; k\rangle$

Scalar field $T(x)$ with negative mass squared called **tachyon**

(sign of instability because $\frac{d^2V(T)}{dT^2} < 0$)

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- $N = 1 = \tilde{N} \longrightarrow M^2 = 0$

States: $\epsilon_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0, 0; k\rangle$ with $k^2 = 0$

$$L_1(\epsilon_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0, 0; k\rangle) \propto (\alpha_0 \cdot \alpha_1 + \dots)(\epsilon_{\mu\nu}\alpha_{-1}^\mu\tilde{\alpha}_{-1}^\nu|0, 0; k\rangle) = 0$$

(the same with \tilde{L}_1)

physical state $\longleftrightarrow k^\mu\epsilon_{\mu\nu} = 0 = k^\nu\epsilon_{\mu\nu}$

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Important: States of the form $|\psi\rangle = L_1^\dagger|\chi\rangle$ are irrelevant because $\langle\text{phys}|L_1^\dagger|\chi\rangle = 0 \quad \forall \quad |\text{phys}\rangle$, meaning that they are orthogonal to all physical states.

At the end what we have is:

Physical states: $\epsilon_{ij} \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0, 0; k\rangle$ with $i, j = 2, \dots, D - 1$

1. Trace: $\epsilon_{ij} \propto \delta_{ij} \longrightarrow$ spinless particle;

1 state, **scalar field** called the dilaton $\phi(x)$

2. Symmetric (traceless) part: $\epsilon_{(ij)} \longrightarrow$ spin 2 particle;

$\frac{(D-2)(D-1)}{2} - 1$ states; **the graviton** $h_{\mu\nu}(x)$ ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$)

3. Antisymmetric part: $\epsilon_{[ij]}$

$\frac{(D-2)(D-3)}{2}$ states; called Kalb-Ramond field $B_{\mu\nu}(x)$

• $N = \tilde{N} \geq 2 \longrightarrow M^2 \geq \frac{4}{l_s^2}$ Very heavy!!!

And what about the open string quantization?

Open string quantization

Again, want to solve $\partial^2 X^\mu(\tau, \sigma) = 0$, now with Neumann b.c.

$$\text{i.e., } \partial_\sigma X^\mu(\tau, 0) = \partial_\sigma X^\mu(\tau, \pi) = 0 \quad \forall \tau$$

As a consequence of b.c. : $\alpha_n^\mu = \tilde{\alpha}_n^\mu \quad \forall n$ (stationary wave)

The solution to the e.o.m. :

$$X^\mu(\tau, \sigma) = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma$$

The quantization process is exactly **the same** as for the closed string

(But just with one set of oscillators)

Let's jump directly to the spectrum...

The constraints translate into:

$$\text{Open string} \left\{ \begin{array}{l} (L_0 - 1)|\psi\rangle = 0 \\ L_{m>0}|\psi\rangle = 0 \end{array} \right.$$

Using the first condition:

$$M^2 = \frac{N - 1}{l_s^2}$$

- $N = 0 \longrightarrow M^2 = -\frac{1}{l_s^2}$ State: $|0; k\rangle$

Open string tachyon field $t(x)$

- $N = 1 \longrightarrow M^2 = 0$ States: $\epsilon_\mu \alpha_{-1}^\mu |0; k\rangle$

If $\epsilon_\mu k^\mu = 0 \longrightarrow$ physical state: $\epsilon_i \alpha_{-1}^i |0; k\rangle$ with $i = 2, \dots, D - 1$

$(D - 2)$ states of a spin 1 particle \longleftrightarrow Massless vector field $A_\mu(x)$

To sum up:

Closed string spectrum:

- Tachyon field $T(x)$
- Dilaton $\phi(x)$
- Graviton $h_{\mu\nu}(x)$
- Kalb-Ramond field $B_{\mu\nu}(x)$
- Infinite tower of massive fields (very heavy!)

Open string spectrum:

- Tachyon field $t(x)$
- Maxwell field $A_\mu(x)$
- Scalar fields (Dirichlet b.c.) $\phi^I(x)$
- Infinite tower of massive fields (very heavy!)

Final comments:

Turns out that to have **no negative norm** states need $D \leq 26$, and in particular with $D = 26$ the spectrum is the same as for an alternative approach where the gauge is completely fixed before quantization (called the light-cone quantization).

Two important problems (at least): we don't want tachyons in the theory, and more importantly there are no fermions!!

To sum up:

Closed string spectrum:

- Tachyon field $T(x)$
- Dilaton $\phi(x)$
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Turns out that to have **no negative norm** states need $D \leq 26$, and in particular with $D = 26$ the spectrum is the same as for an alternative approach where the gauge is completely fixed before quantization (called the light-cone quantization).

This is why we need **SUPERSTRING THEORY**

Before we continue, let me say that by studying how all these fields interact it is possible to construct an effective action.

In particular, for the massless modes of the closed superstring:

$$S = \frac{1}{(2\pi)^7 g_s^2 l_s^8} \int d^{10}x \sqrt{g} \left[R - \frac{1}{2} (\nabla\phi)^2 - \frac{1}{2} e^{-\phi} H_3^2 - \frac{1}{2} e^{2\phi} F_1^2 - \frac{1}{2} e^{\phi} F_3^2 - \frac{1}{4} F_5^2 \right] - \frac{1}{(2\pi)^7 g_s^2 l_s^8} \int C_4 \wedge H_3 \wedge F_3,$$

Note: don't worry about the details...

This is known as the **supergravity action** (and corresponds to the low energy ($E \ll 1/l_s$) limit of Type IIB string theory).

There is also an effective action for the massless modes of the open superstring and we will check it later on.

D-branes

Within string theory, spacetime is only part of a much more complex structure



whose small excitations are strings and whose large, solitonic excitations include what we generically call: **BRANES**



0-brane



1-brane



2-brane



3-brane

with masses $m \propto 1/g_s$ or $m \propto 1/g_s^2$, i.e., **very heavy!**

We will center our attention on solitons of the first kind, known as:

D-BRANES

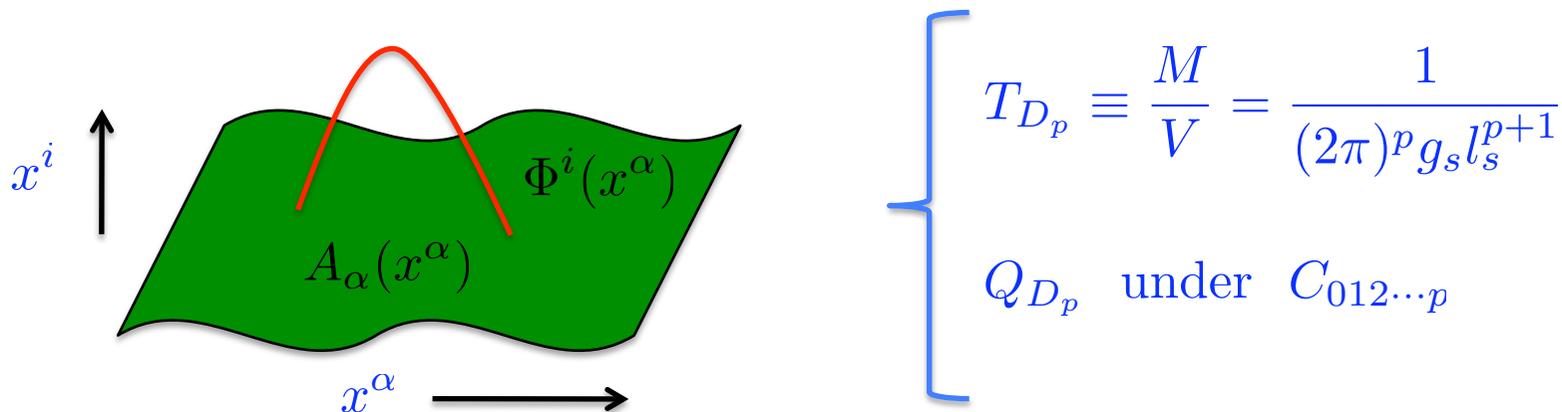
Remember b.c.:

$$X^i(\tau, 0) = X^i(\tau, \pi) = c^i \quad i = p + 1, \dots, D - 1 \quad (\text{Dirichlet: fixed endpoints})$$

$$\partial_\sigma X^\alpha(\tau, 0) = \partial_\sigma X^\alpha(\tau, \pi) = 0 \quad \alpha = 0, \dots, p \quad (\text{Neumann: free endpoints})$$

From the quantization of the open string one finds that:

- A D-brane has a **Maxwell field**, and **massless scalar field** for each normal direction, living on its world volume.
- The scalar fields represent the fluctuations of the D-brane in the transverse directions.



Remember b.c.:

$$X^i(\tau, 0) = X^i(\tau, \pi) = c^i \quad i = p + 1, \dots, D - 1 \quad (\text{Dirichlet: fixed endpoints})$$

$$\partial_\sigma X^\alpha(\tau, 0) = \partial_\sigma X^\alpha(\tau, \pi) = 0 \quad \alpha = 0, \dots, p \quad (\text{Neumann: free endpoints})$$

From the quantization of the open string one finds that:

- A D-brane has a **Maxwell field**, and **massless scalar field** for each normal direction, living on its world volume.
- The scalar fields represent the fluctuations of the D-brane in the transverse directions.

D-branes are dynamical objects with mass and charge

How do we describe their dynamics?

Again, using a generalized version of the relativistic point particle action, we have:

$$S = -T_{D_p} \int d^{p+1}\xi \sqrt{-\det g_{\mu\nu}} \quad g_{\mu\nu} \equiv \text{induced metric on the world-volume}$$

This action just describes **how the D-brane moves** but doesn't say anything about the Maxwell field.

It turns out that in the low energy limit ($E \ll 1/l_s$) and when spacetime variation of the fields is small:

$$S = -\frac{1}{(2\pi)^p g_s l_s^{p+1}} \int d^{p+1}\xi \sqrt{-\det(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$$

which describes a $U(1)$ gauge theory in $p + 1$ dimensions with $D - p - 1$ scalar fields.

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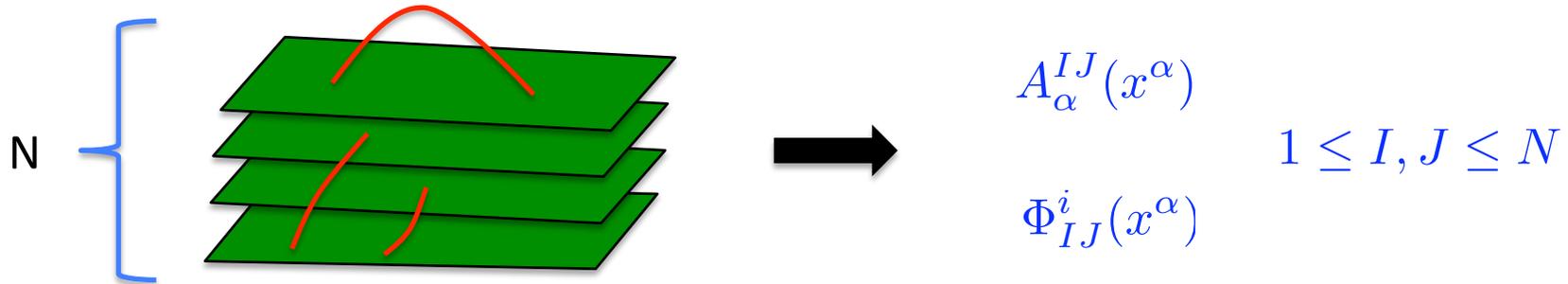
$$S = -\frac{1}{(2\pi)^p g_s l_s^{p+1}} \int d^{p+1}\xi \sqrt{-\det(g_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}$$

and if $g_{\mu\nu} = \eta_{\mu\nu}$ then

$$S \simeq -\frac{1}{(2\pi)^p g_s l_s^{p+1}} \int d^{p+1}\xi \left(1 + \frac{(2\pi l_s^2)^2}{4} F^2 + (2\pi l_s^2)^4 F^4 + \dots + \right)$$

Maxwell's lagrangian

And with N parallel D p -branes?



$\mathcal{N} = 2^{(7-p)/2}$ Super Yang-Mills (SYM) theory in $D = p + 1$ dimensions with $SU(N)$ gauge group and $g_{\text{YM}}^2 = (2\pi)^{p-2} g_s l_s^{p-3}$

In particular, for $p = 3$, which is the case we will be interested in, the low energy theory is:

$\mathcal{N} = 4$ Super Yang-Mills (SYM) theory in (3+1) dimensions with $SU(N)$ gauge group

We have mentioned that D-branes are very heavy objects, so we would expect these branes to deform the spacetime.

Black p-branes

Extended p-dimensional versions of charged black holes.

For low energies, a black p-brane can be described as a solution to the supergravity equations of motion.

Recall: In 4 dimensions, if you want a static and spherically symmetric solution you obtain the Schwarzschild black hole. If you coupled gravity to a U(1) gauge field you will get the Reissner-Nordström black hole.

Let's focus only on p=3 (because that is what we will need later)

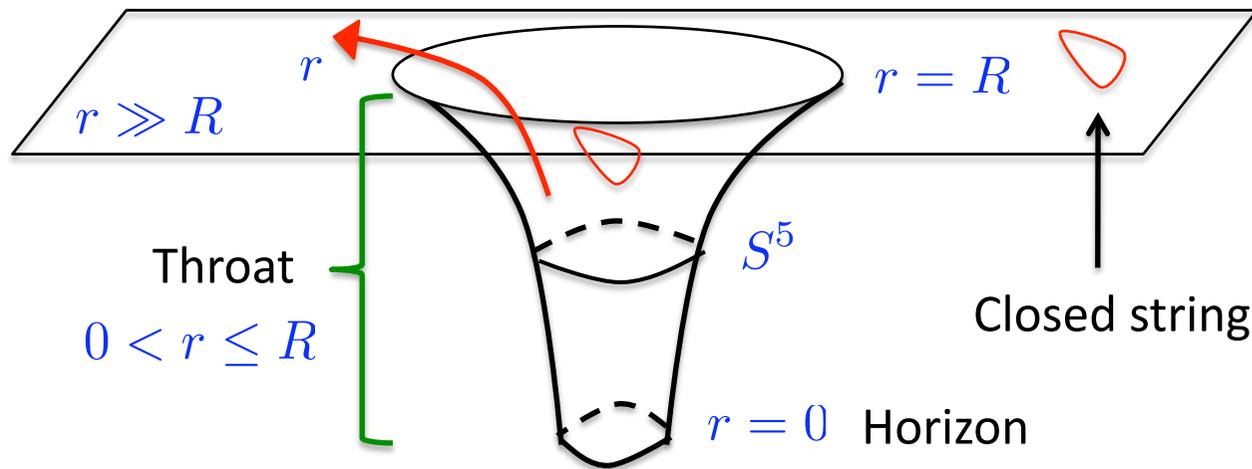
We are looking for a solution of the e.o.m. that is translationally and rotationally invariant in 3 spatial directions and is charged under C_{0123}

The extremal black 3-brane solution

The metric:

$$ds^2 = f^{-\frac{1}{2}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2)$$

with $f = 1 + \frac{R^4}{r^4}$ and $R^4 = 4\pi g_s \alpha'^2 N$



Relation between the mass and the charge: $M \geq \frac{N}{(2\pi)^3 g_s l_s^{p+1}}$ (BPS bound)

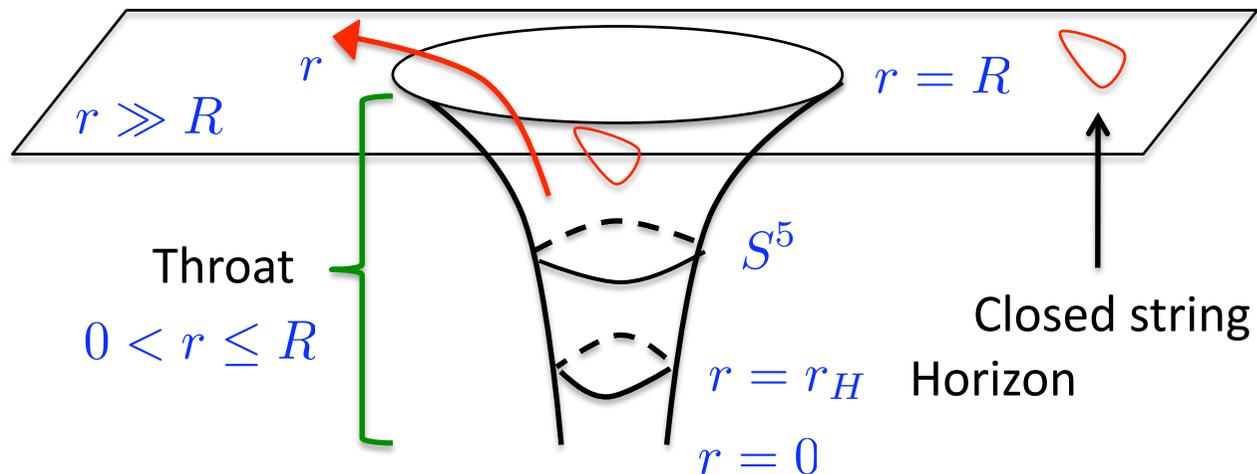
Extremal: $M = \frac{N}{(2\pi)^3 g_s l_s^{p+1}}$ \rightarrow Same as stack on N D3-branes!

The non-extremal black 3-brane solution

The metric:

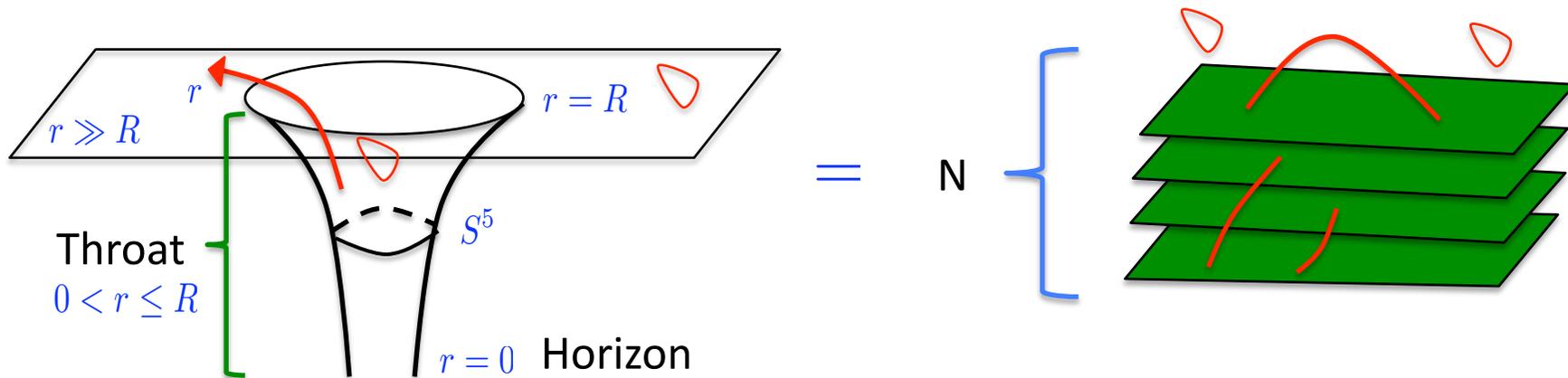
$$ds^2 = f^{-\frac{1}{2}} (-h dt^2 + d\vec{x}^2) + \frac{f^{\frac{1}{2}}}{h} (dr^2 + r^2 d\Omega_5^2)$$

with $f = 1 + \frac{R^4}{r^4}$ and $h = 1 - \frac{r_H^4}{r^4}$



From now on, let's just focus on the extremal solution

Two alternative descriptions of the same object?



Under perturbative control

when: $R/l_s \gg 1$ and $g_s \ll 1$

which can be rewritten: $g_s N \gg 1$

$$(R^4 = 4\pi\Gamma(2)g_s N l_s^4)$$

Under perturbative control

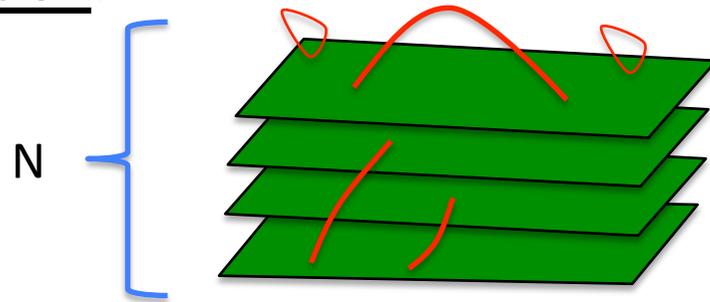
when: $g_s N \ll 1$ and $g_s \ll 1$

Let's investigate the low energy limit of these two descriptions

The AdS/CFT correspondence

[Maldacena (1997); Gubser, Klebanov, Polyakov; Witten (1998)]

First description:



Low energy limit: $E \ll 1/l_s$
($l_s \rightarrow 0$ and E fixed)

$$S = S_{sugra} + S_{brane} + S_{int}$$

$S_{sugra} \equiv$ describes the massless modes of the closed strings

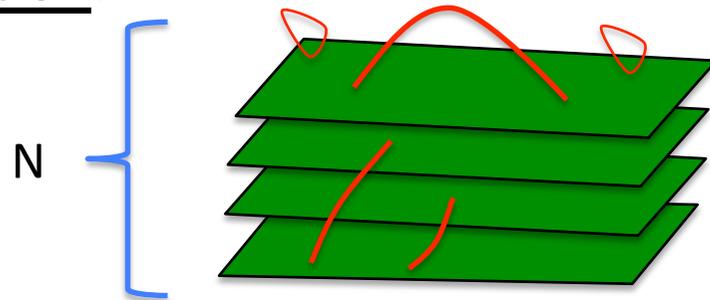
$S_{brane} \equiv$ describes the excitations of the $D3$ -branes which corresponds to $\mathcal{N} = 4$ SYM in (3+1) dimensions

$S_{int} \equiv$ describes the interactions between closed and open strings

The AdS/CFT correspondence

[Maldacena (1997); Gubser, Klebanov, Polyakov; Witten (1998)]

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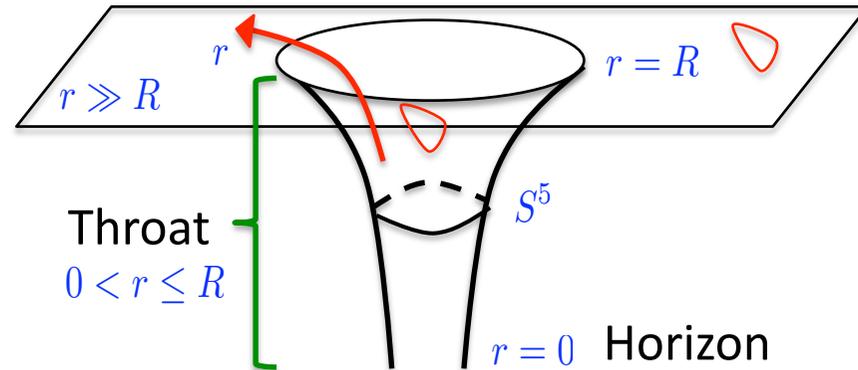
$$S = S_{sugra} + S_{brane} + S_{int}$$

Turns out that in this limit the open and the closed strings no longer interact with each other. So we obtain **two decoupled systems**:

Free supergravity in flat (9+1) spacetime + $\mathcal{N} = 4$ SYM in (3+1)

Second description:

Low energy limit: $E \ll 1/l_s$
($l_s \rightarrow 0$ and E fixed)



$$ds^2 = f^{-\frac{1}{2}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + f^{\frac{1}{2}} (dr^2 + r^2 d\Omega_5^2)$$

with $f = 1 + \frac{R^4}{r^4}$ and $R^4 = 4\pi g_s \alpha'^2 N$

Something very important about this metric: g_{tt} is not constant, so, there is a non trivial relation between the energy measured by an observer at infinity and the energy at a fixed value of the coordinate r :

$$E = \left(1 + \frac{R^4}{r^4}\right)^{-\frac{1}{4}} E_r$$

Therefore, if we take an object very close to $r = 0$, the observer at infinity will measure very **small energies**

Now, let's consider the low energy limit for two regions:

- **Far from the horizon**: Just as before, we have free sugra in flat spacetime
- **Close to the horizon**: due to the red shift effect, we have strings with arbitrary local energies

Due to gravitational potential, the string modes of these two regions can not interact with each other. So again, we obtain **two decoupled systems**:

Free supergravity in
flat (9+1) spacetime

+

Type IIB string theory in
the region close to the
horizon

Therefore, if we take an object very close to $r = 0$, the observer at infinity will measure very **small energies**

Now, let's consider the low energy limit for two regions:

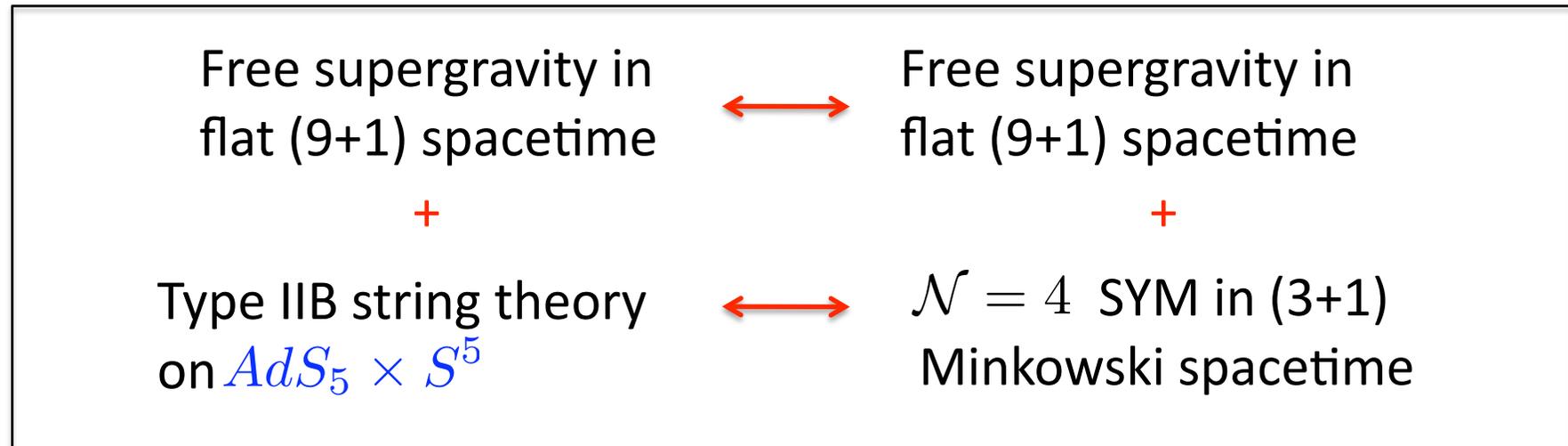
- **Far from the horizon**: Just as before, we have free sugra in flat spacetime
- **Close to the horizon**: due to the red shift effect, we have strings with arbitrary local energies

But, what is the region close to the horizon?

If $r \ll R$ then $f \sim \frac{R^4}{r^4}$ and we can write:

$$ds^2 = \underbrace{\frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)}_{AdS_5} + \underbrace{\frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2}_{S^5}$$

Now, the final conclusion, we have **two equivalent ways** to describe the same physical system, i.e., a stack of N D3-branes:



So, the conjecture is that:

Type IIB string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ SYM on (3+1)-dim Minkowski spacetime

$SU(N)$ $\mathcal{N} = 4$ SYM on flat 3+1 dimensions $=$ Type IIB string theory on $AdS_5 \times S^5$ spacetime

Field content:

$$A^\mu(x)$$

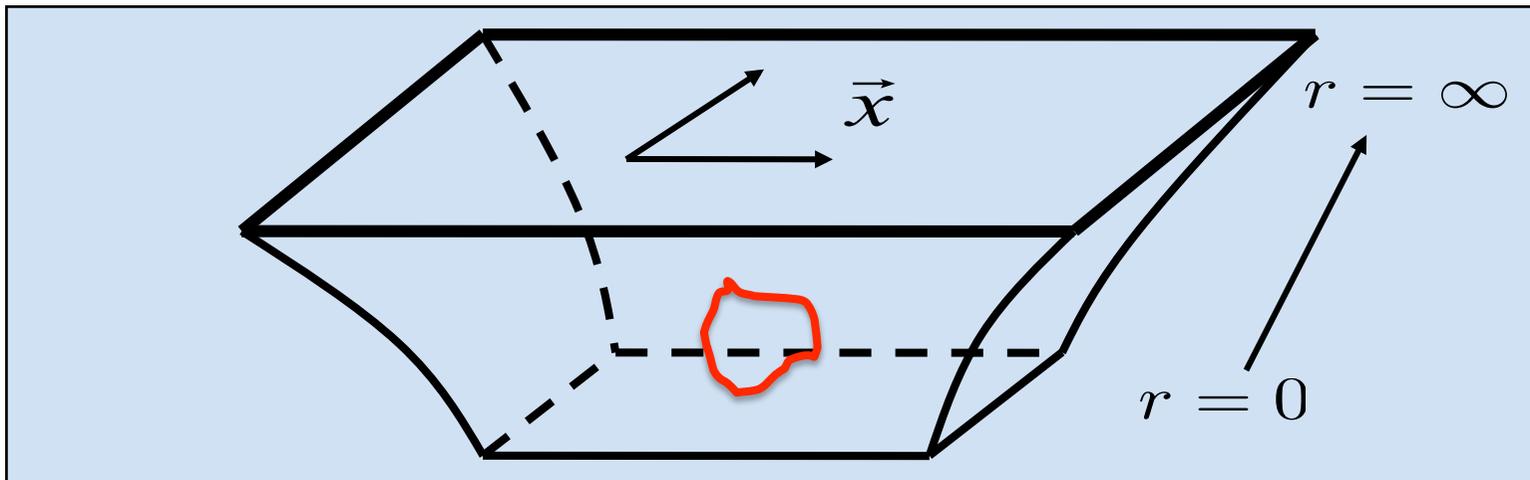
$$\Phi^I(x) \quad I = 1, \dots, 6$$

$$\lambda^a(x) \quad a = 1, \dots, 4$$

All massless fields and in the adjoint rep.

$SU(N)$ $\mathcal{N} = 4$ SYM = Type IIB string theory on
on flat 3+1 dimensions $AdS_5 \times S^5$ spacetime

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2$$



IMPORTANT

Remember: String theory calculations are under control when the coupling is weak and the curvature is small, i.e.

$$g_s \ll 1 \quad \text{and} \quad R/l_s \gg 1$$

which implies:

$$N \gg 1 \quad \text{and} \quad g_{\text{YM}}^2 N \gg 1$$

i.e., we can study the gauge theory in the **large N** and **strong coupling limit**.

IMPORTANT

Remember: String theory calculations are under control when the coupling is weak and the curvature is small, i.e.

$$g_s \ll 1 \quad \text{and} \quad R/l_s \gg 1$$

which implies:

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Using the AdS/CFT correspondence we can obtain useful information about **SYM** (and some other strongly coupled gauge theories) doing simple calculations in **classical string theory on a curved background**.

$SU(N)$ $\mathcal{N} = 4$ SYM on flat 3+1 dimensions = Type IIB string theory on $AdS_5 \times S^5$ spacetime

There is a dictionary (still under construction):

- | | |
|------------------------------|--|
| • N and g_{YM} | • g_s and R/l_s |
| • (t, \vec{x}) | • (t, \vec{x}) |
| • Energy scale | • r |
| • Internal space | • $(\theta_1, \theta_2, \theta_3, \theta_4)$ |
| • Conformal group $SO(4, 2)$ | • Isometry group $SO(4, 2)$ (AdS_5) |
| • Internal symmetry $SO(6)$ | • Isometry group $SO(6)$ (S^5) |

Geometry on RHS is dynamical, pure AdS_5 correspond to SYM vacuum. Excitations on top of AdS_5 correspond to other SYM states.

$$\langle \text{Tr}[P \exp(i \oint dx^\mu A_\mu(x))] \rangle = \exp[iS[X(r = \infty)]]$$

How different is SYM from QCD?

$\mathcal{N} = 4$ SYM

$T = 0$

QCD

- g_{YM} does not run
- Deconfined
- Conformal (scale-invariant)
- SUSY
- Matter in the adjoint rep.

- $g_{\text{YM}} = g_{\text{YM}}(E)$
- Confined at low energies
- Non-conformal
- No SUSY
- Matter in the fundamental

But...

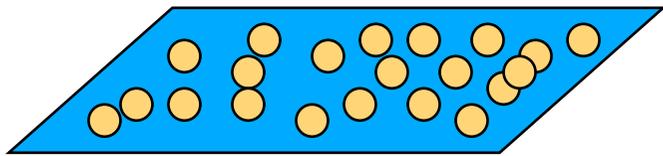
$T > T_c$

- $\epsilon \propto T^4$
- Still deconfined
- SUSY broken
- Non-abelian plasma made of gluons and matter in the adjoint.

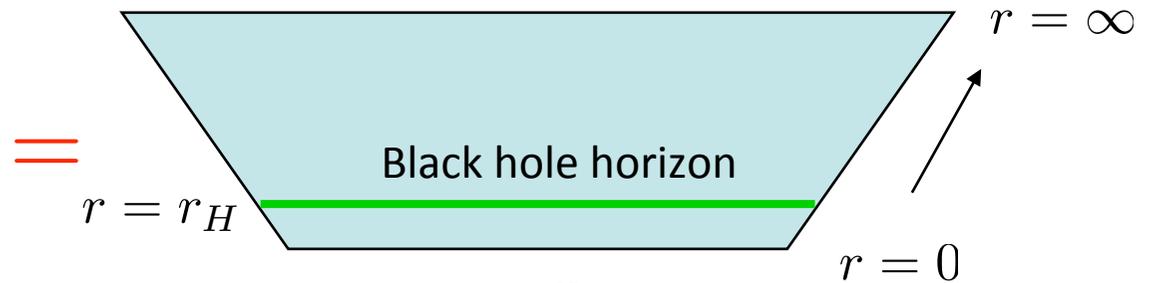
- $\epsilon \propto T^4$
- Deconfined
- No SUSY
- Non-abelian plasma made of quark and gluons

How do we turn on the temperature?

$\mathcal{N} = 4$ SYM at finite temperature = Type IIB string theory on AdS_5 with a black hole [Witten]



Plasma of gluons and matter in the adjoint



$$T = \frac{r_H}{\pi R^2}$$

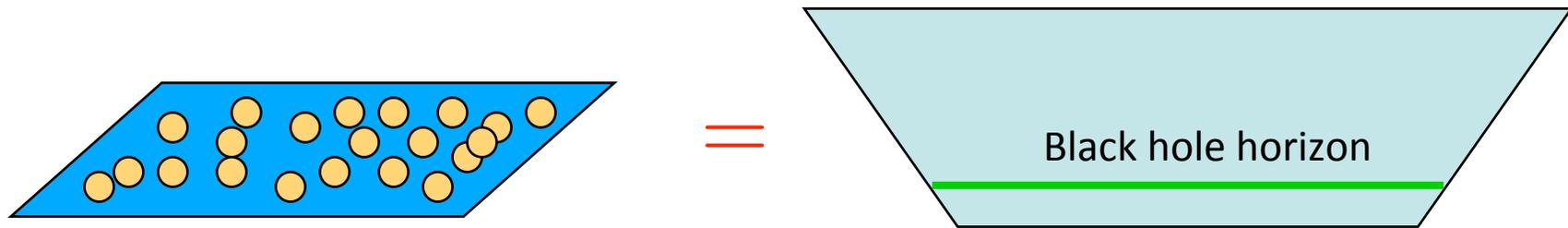
The metric:

$$ds^2 = \frac{1}{\sqrt{H}} (-h dt^2 + d\vec{x}^2) + \frac{\sqrt{H}}{h} dr^2 + R^2 d\Omega_5^2$$

with $H = \frac{R^4}{r^4}$ and $h = 1 - \frac{r_H^4}{r^4}$

Applications of AdS/CFT

1- The entropy of SYM plasma



According to dictionary:

$$S_{plasma} = S_{BH} = \frac{A_H}{4G_N}$$

At weak coupling:

$$S_{plasma} = \frac{2\pi^2}{3} N^2 T^3 V$$

At strong coupling:

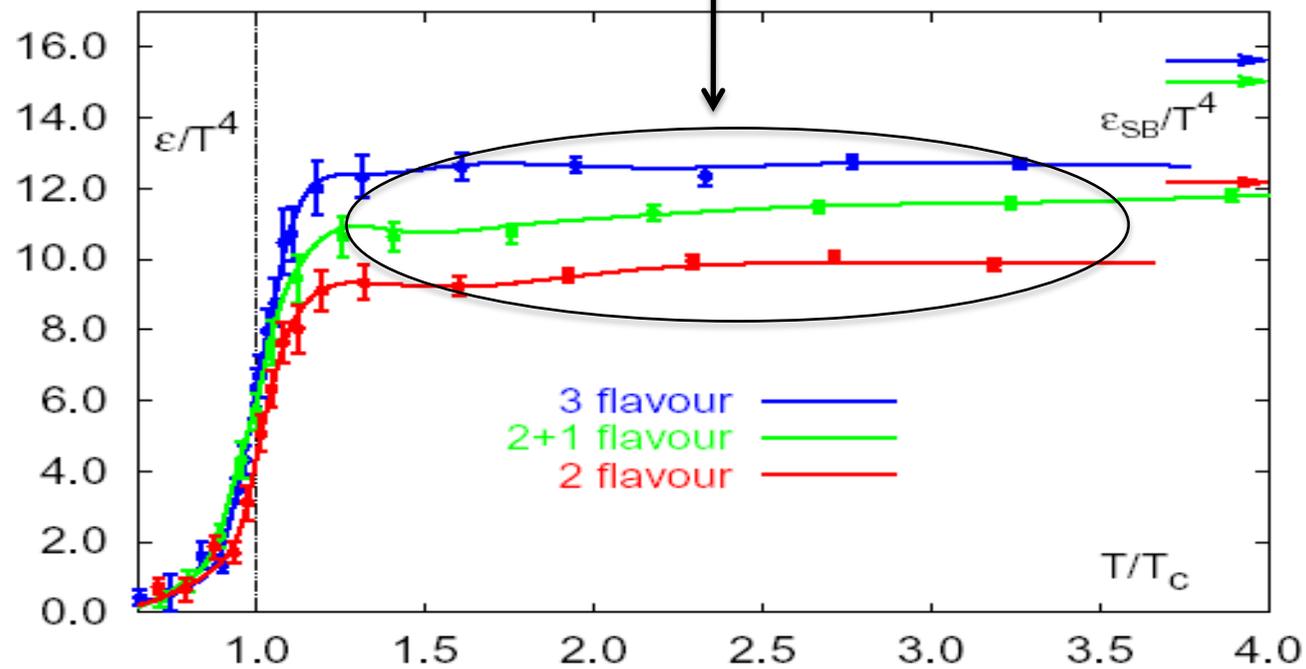
$$S_{plasma} = \frac{2\pi^2}{3} N^2 T^3 V \left(\frac{3}{4}\right)$$

[Gubser, Klebanov, Peet]

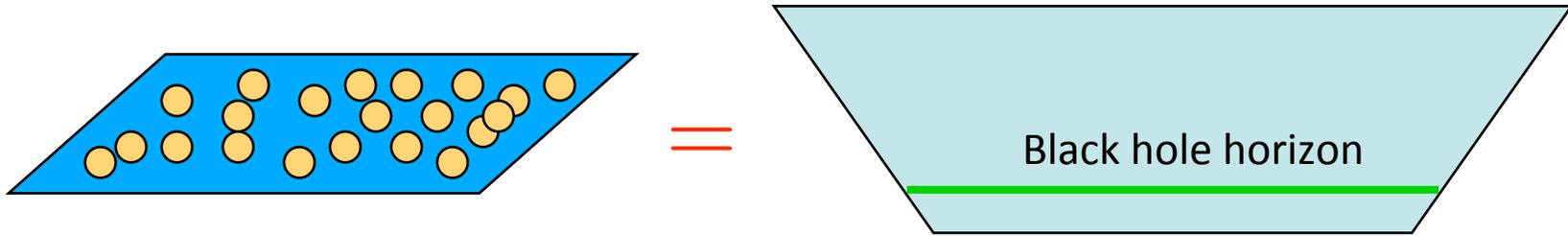
Then, $S_{\text{plasma}}^{\text{strong}} = 0.75 S_{\text{plasma}}^{\text{weak}}$, which is similar to the results

from lattice QCD at $T \sim (1 - 4)T_c$:

Energy and entropy densities $\sim 0.8-0.85$ of an ideal gas



2-The shear viscosity of SYM plasma



$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int d^4x e^{i\omega t} \langle [T_{xy}(x), T_{xy}(0)] \rangle = \lim_{\omega \rightarrow 0} \frac{1}{16\pi G_N} \sigma_{h_{\mu\nu}}(\omega)$$

Kubo's formula

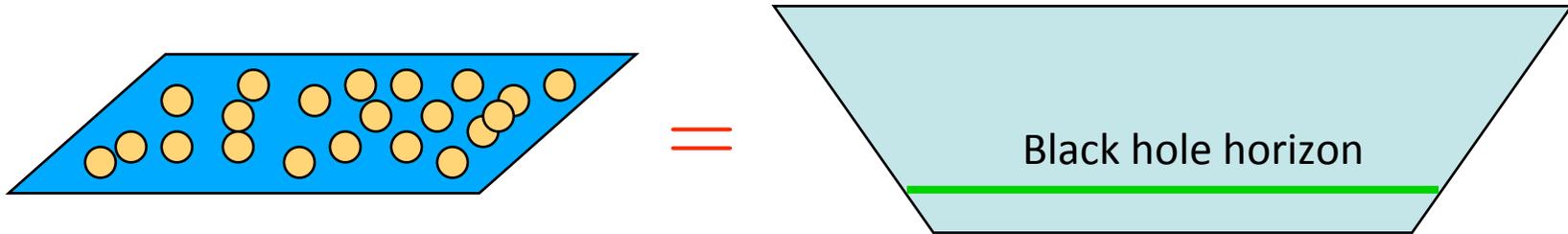
AdS/CFT formula

[Callan; Gubser, Klebanov, Polyakov; Witten]

At weak coupling: $\frac{\eta}{s} \sim \frac{1}{(g_{\text{YM}}^2 N)^2 \log(1/g_{\text{YM}}^2 N)} \gg 1$ [Arnold, Moore, Yaffe]

At strong coupling: $\frac{\eta}{s} \sim \frac{1}{4\pi}$ [Policastro, Son, Starinets]

2-The shear viscosity of SYM plasma



$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int d^4x e^{i\omega t} \langle [T_{xy}(x), T_{xy}(0)] \rangle = \lim_{\omega \rightarrow 0} \frac{1}{16\pi G_N} \sigma_{h_{\mu\nu}}(\omega)$$

Kubo's formula

AdS/CFT formula

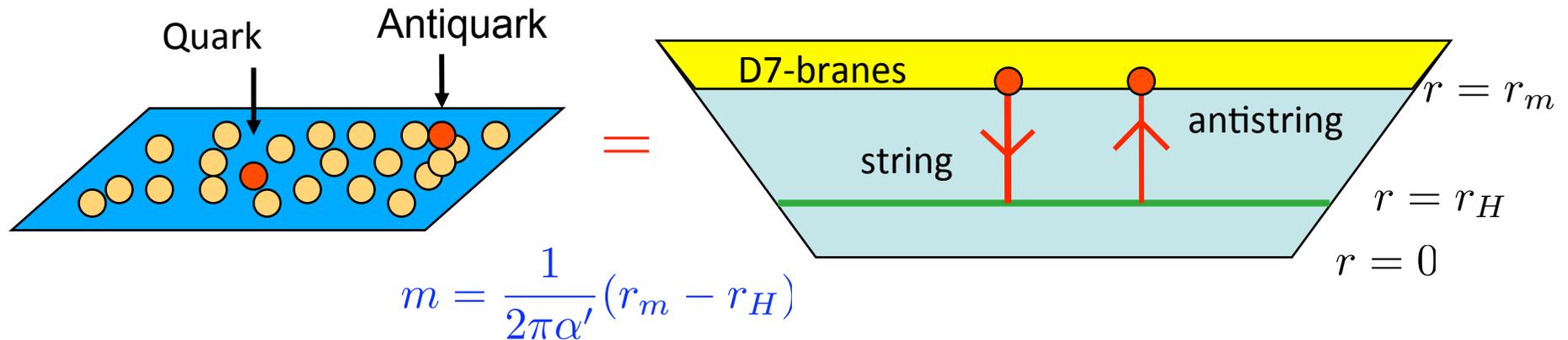
[Callan; Gubser, Klebanov, Polyakov; Witten]

At weak coupling: $\frac{\eta}{s} \sim \frac{1}{(g_{\text{YM}}^2 N)^2 \log(1/g_{\text{YM}}^2 N)} \gg 1$ [Arnold, Moore, Yaffe]

At strong coupling: $\frac{\eta}{s} \sim \frac{1}{4\pi} \left(1 + \frac{135\zeta(3)}{8\lambda^{3/2}} + \dots \right) < 1$ [Policastro, Son, Starinets; Buchel et al. 2005]

How do we add matter in the fundamental?

[Karch, Katz]



- The **string endpoint** represents **the quark**, while the rest of the string codifies information about the gluonic fields.
- The **quark velocity** corresponds to the **coordinate velocity** of the string endpoint. The relation between proper and coordinate velocity is given by:

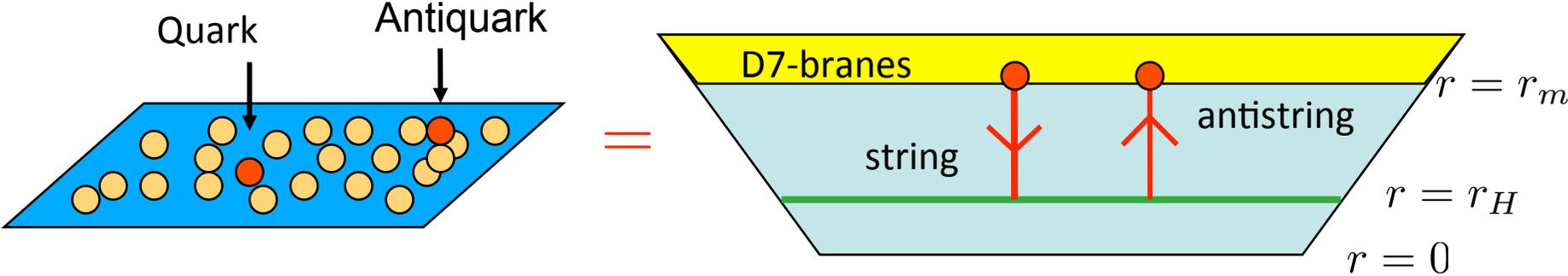
$$V_{prop} = \frac{v}{\sqrt{1 - (r_H^4/r_m^4)}} \leq 1$$

So AdS/CFT implies a bound for the quark velocity:

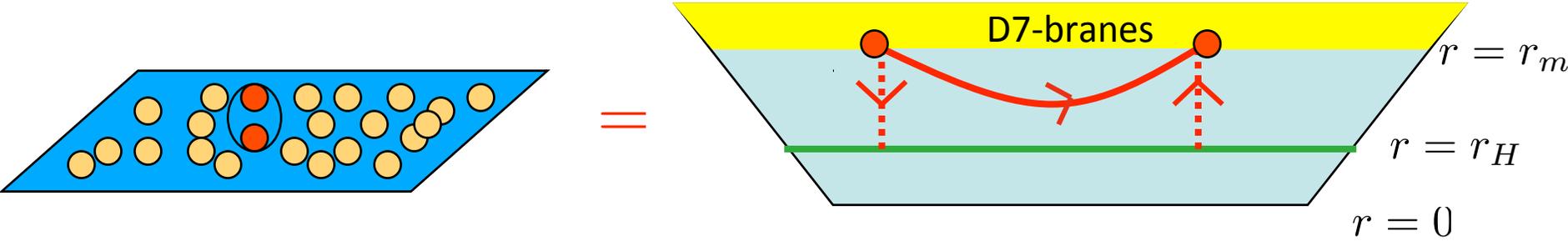
$$v \leq v_m \equiv \sqrt{1 - (r_H^4/r_m^4)} \approx 1 - \frac{\sqrt{g_{YM}^2 NT}}{2m_q}$$

[Argyres, Edalati, Vázquez-Poritz]

How do we add matter in the fundamental?

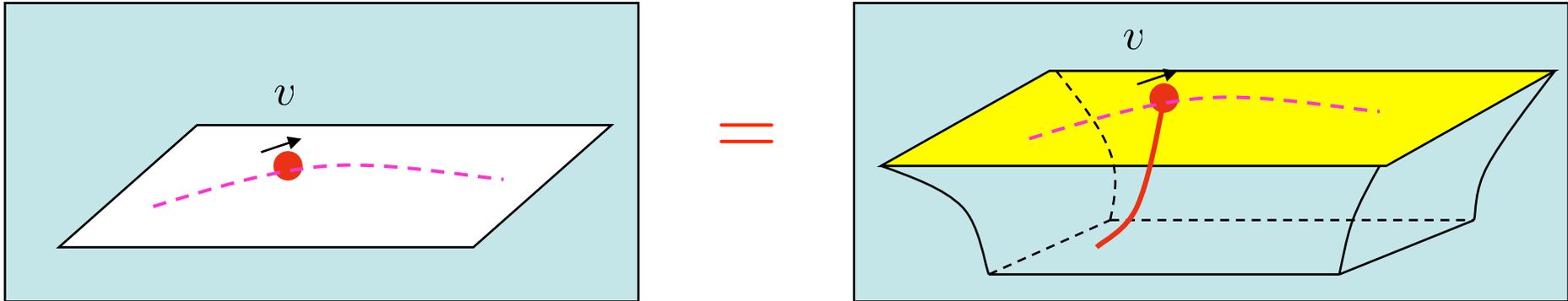


And to represent a quark-antiquark pair :



3- Energy loss: zero temperature

Infinitely massive quark:



$$E(t) = \frac{\sqrt{g_{\text{YM}}^2 N}}{2\pi} \int_{-\infty}^t dt_{\text{ret}} \frac{\vec{a}^2 - [\vec{v} \times \vec{a}]^2}{(1 - v^2)^3} + \gamma m_q$$

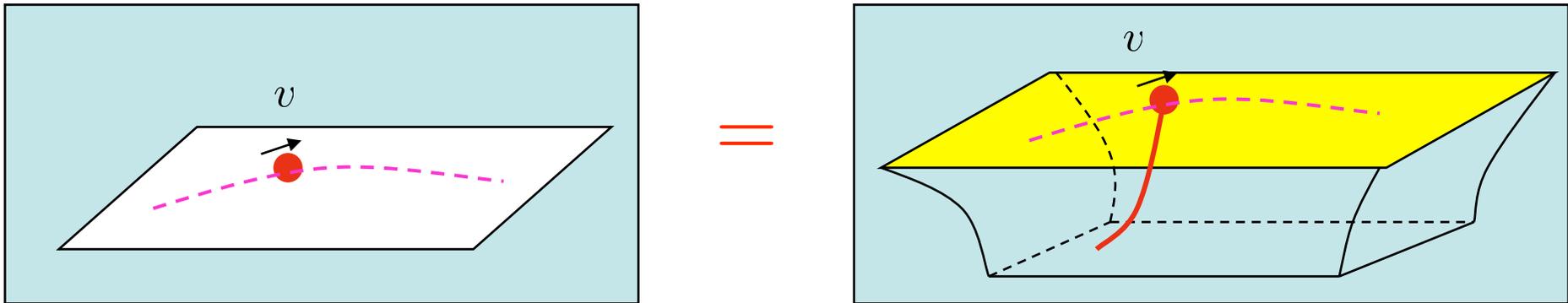
[Mikhailov]

↑
Lienard's formula!

↑
Intrinsic energy of the quark

3- Energy loss: zero temperature

Massive quark:



$$E(t) = \frac{\sqrt{g_{\text{YM}}^2 N}}{2\pi} \int_{-\infty}^t dt F^2 \left(\frac{2\pi m_q^2 - \sqrt{g_{\text{YM}}^2 N} v F}{(2\pi)^2 m_q^2 - (g_{\text{YM}}^2 N) F^2} \right) + \left(\frac{2\pi m_q^2 - \sqrt{g_{\text{YM}}^2 N} v F}{\sqrt{(2\pi)^2 m_q^2 - (g_{\text{YM}}^2 N) F^2}} \right) \gamma m_q$$

↑
Energy loss

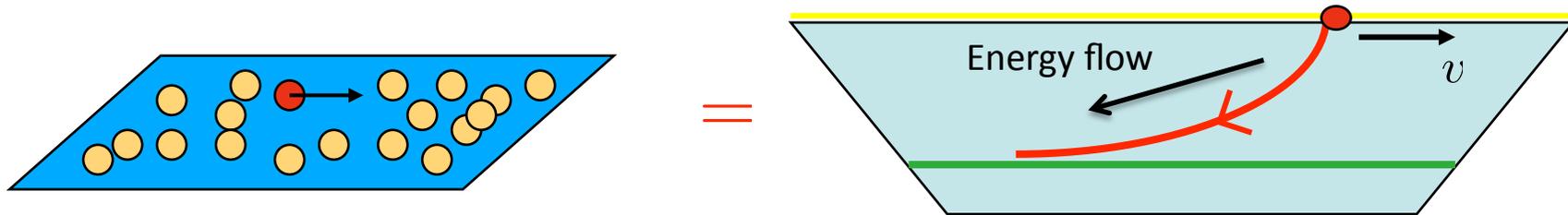
↑
Intrinsic energy of the quark

$F \equiv$ External force

[MCh, Güijosa]

4- Energy loss: finite temperature

A heavy external quark moving at speed v experience a drag force



Solving the classical e.o.m for the string it is easy to obtain:

$$\frac{dE}{dt} = -\frac{\pi\sqrt{g_{\text{YM}}^2 N}}{2} T^2 \frac{v^2}{\sqrt{1-v^2}} \quad \text{and} \quad \frac{dp}{dt} = -\frac{\pi\sqrt{g_{\text{YM}}^2 N}}{2} T^2 \frac{v}{\sqrt{1-v^2}}$$

[Casalderrey, Teaney; Herzog, Kovtun, Karch, Kozcaz, Yaffe; Gubser]

Using $\mu = -\frac{\dot{p}}{p}$ we can estimate the friction coefficient:

$$\mu = \frac{\pi\sqrt{g_{\text{YM}}^2 N} T^2}{2m} \left\{ \begin{array}{ll} \text{Charm} & \begin{array}{ll} \text{AdS/CFT} & \text{pQCD} \\ \mu \approx 1.6 \text{ c/fm} & \mu \approx 0.22 \text{ c/fm} \\ \text{[Gubser]} & \text{[van Hees, Greco, Rapp]} \end{array} \end{array} \right.$$

$$(m_c = 1.4\text{GeV}, N = 3, g_{\text{YM}}^2 N = 6\pi, T = 250\text{MeV})$$

Conclusions

Just about the AdS/CFT correspondence:

- The AdS/CFT correspondence seems to be a useful tool to obtain information about certain strongly-coupled gauge theories.
- The QGP produced at RHIC (and soon at the LHC) appears to be the most promising site for string theory to make contact with the real world.
- A lot remains to be done. Some examples: we need to find the string theory dual to QCD (might be very hard), refine our descriptions, e.g., finite size and time dependent plasma, initial stage, etc.

Final comment

I strongly recommend that you attend to:

- **Samuel Vazquez's** talk: "On a consistent AdS/CFT description of boost invariant plasma" (Monday afternoon)
- **Alex Buchel's** talks: "Hydrodynamics in non-conformal gauge theories" and "Conformal hydrodynamics beyond the supergravity approximation" (Monday)
- **José Edelstein's** talks: "The AdS/CFT correspondence and non-perturbative QCD" and "Jet quenching in heavy ion collisions from AdS/CFT" (Monday and Tuesday)
- **Brian Wecht's** talk: "Compactification with torsion: moving beyond Calabi-Yau" (Tuesday afternoon)

Thank you