Linear response of entanglement entropy from holography

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Based on:

arXiv:1602.05934, Sandipan Kundu & JFP
arXiv:1705.10324, Sagar Lokhande, Gerben Oling & JFP

Outline

1. Motivation
2. Models of global quenches in holography
3. Entanglement entropy after instantaneous quenches
   - Spread of entanglement in $(1+1)-$dimensional CFTs
   - Perturbative computation in higher dimensions
4. Linear response of entanglement entropy
   - Fefferman-Graham expansion
   - Entanglement entropy after general global quenches
   - General properties and examples
5. Conclusions

Motivation: Quenches

Objective:

Characterize generic out-of-equilibrium states in AdS/CFT

- Are there universal rules that govern the evolution?
- What is the behavior of the various observables?
- What is the nature of the final state?

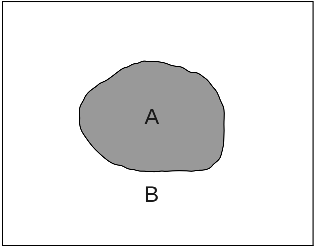
Simplest dynamical process is a quantum quench:

$$H_\lambda = H_0 + \lambda(t,\vec{x})\,\delta H_\Delta \quad \Longrightarrow \quad \mathcal{L}_\lambda = \mathcal{L}_0 + \lambda(t,\vec{x})\,\mathcal{O}_\Delta$$

$$\text{Outcome:} \begin{cases} \text{Thermalization: } \rho(t) \to \rho_{\text{thermal}} + \mathcal{O}(e^{-S}) \\ \text{Relaxation: } \rho(t) \to \rho_{\text{initial}} \\ \text{Quantum revivals: } \rho(t) \to \rho(t - t_{\text{p}}) \end{cases}$$

Motivation: Observables

- One-point functions of local operators $\mathcal{O}_i$ thermalize fast
- A useful order parameter is entanglement entropy S_A



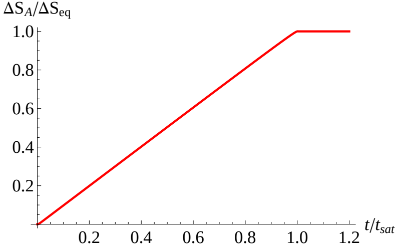
Hilbert space factorizes $\mathcal{H}_{\text{total}} = \mathcal{H}_{\text{A}} \otimes \mathcal{H}_{\text{B}}$, then define $\rho_A \equiv \text{tr}_B[\rho]$

$$\boxed{S_A = -\text{tr}[\rho_A \log \rho_A]}$$

Review of the 'entanglement tsunami' proposal

[Calabrese & Cardy] showed that for weakly coupled $(1+1)$ CFTs:

$$\Delta S_A(t) = 2t\, s_{\text{eq}}\,, \qquad t \leq t_{\text{sat}} = R$$



- Instantaneous quenches: $\lambda(t) \sim \delta(t) \leftrightarrow \langle T_{00}(t) \rangle \sim \theta(t)$
- Large subsystems: $R \gg 1/T$
- Explanation in terms of free streaming EPR pairs

Review of the ‘entanglement tsunami’ proposal

Q: How do (strong) interactions affect this result?

Early numerical explorations [Abajo-Arrastia et.al, Balasubramanian et.al] and analytical work [Hartman & Maldacena, Liu & Suh] showed that:

$$\Delta S_A(t) = v_E s_{\text{eq}} A_\Sigma t\,, \qquad t_{\text{loc}} \ll t \ll t_{\text{sat}}$$

$$\boxed{v_E = \sqrt{\frac{d}{d-2}}\left(\frac{d-2}{2(d-1)}\right)^{\frac{d-1}{d}}}$$

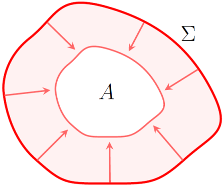


Figure: Pictorial representation of the entanglement tsunami of region A.

Review of the 'entanglement tsunami' proposal

Comments:

- $v_E = 1$ in $d = 2$, as for weakly-coupled theories!
  - EE for multiple strips differs for $d = 2$ [Asplund & Bernamonti]
  - Quasiparticle description fails at large-c [Asplund et.al]
  - In higher dimensions [Casini, Liu & Mezei]

$$v_E^{\text{free}} = \frac{\Gamma[\frac{d-1}{2}]}{\sqrt{\pi}\Gamma[\frac{d}{2}]} \leq v_E$$

- $v_E \leq 1$ suggests a causality bound
  - [Liu & Suh] conjectured that

$$\mathfrak{R}(t) \equiv \frac{1}{s_{\text{eq}} A_\Sigma} \frac{dS_A}{dt} \leq v_E \leq 1$$

  - Proof by [Casini, Liu & Mezei] and [Hartman & Afkhami-Jeddi]
  - Bound fails for small subsystems! No linear growth [Kundu & Pedraza]

Models of global quenches in holography

- In QFT: deform the theory by an operator $\mathcal{O}_\Delta$ with an homogeneous time dependent coupling $\lambda(t)$
- In AdS: turn on the non-normalizable mode of the field dual to $\mathcal{O}_\Delta$
- Electric field quench in $(3+1)$ is analytically tractable (!)

$$S = \frac{1}{2\kappa^2}\int d^{3+1}x\sqrt{-g}\left(R - 2\Lambda - F^2\right)$$

A solution for an arbitrary $E(v)$ is: [Horowitz, Iqbal & Santos]

$$\begin{aligned} ds^2 &= \frac{1}{z^2}\left(-f(v,z)dv^2 - 2dvdz + dx^2 + dy^2\right) \\ F &= -E(v)dv\wedge dx \end{aligned}$$

where

$$f(v,z) = 1 - z^3 m(v)\,, \quad m(v) = \frac{1}{2}\int_{-\infty}^{v} E(v')^2 dv'$$

Models of global quenches in holography

- Another example is a scalar field quench:

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}\left(R - 2\Lambda - \frac{1}{2}(\partial\phi^2)\right)$$

Tractable perturbatively: [Bhattacharyya & Minwalla]

$$\begin{aligned} ds^2 &= \frac{1}{z^2}\left[-f(v,z)dv^2 - 2dvdz + g(v,z)(dx^2+dy^2)\right] \\ \phi &= \phi(v,z) \end{aligned}$$

where, for any $\phi_0(v)$:

$$\begin{aligned} f(v,z) &= 1 + \left(\frac{3}{4}z^2\dot{\phi}_0^2 - z^3 m(v)\right)\epsilon^2 + \cdots\,, \quad m(v) = -\frac{1}{2}\int_{-\infty}^{v} dt\,\dot{\phi}_0\dddot{\phi}_0 \\ g(v,z) &= 1 - \frac{1}{4}z^2\dot{\phi}_0^2\epsilon^2 + \cdots\,, \qquad \phi(v,z) = (\phi_0 + z\dot{\phi}_0)\epsilon + \cdots \end{aligned}$$

Observables are rather insensitive to transients [Joshi et.al.]

Models of global quenches in holography: AdS-Vaidya

- A general theory:

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \left(R - 2\Lambda + \mathcal{L}_{matter} \right)$$

admits an AdS-Vaidya solution:

$$ds^2 = \frac{1}{z^2}\left[-f(v,z)dv^2 - 2dvdz + d\vec{x}^2\right]\,, \quad f(v,z) = 1 - z^d m(v)$$

provided that the stress-tensor is made of null dust:

$$T_{\mu\nu} = \frac{d-1}{4\kappa^2} z^{d-1} \frac{dm}{dv} \delta^v_\mu \delta^v_\nu \tag{1}$$

- Limitations:
  - Phenomenological approach: source is not known
  - Is any $m(v)$ physically reasonable? $\rightarrow$ NEC requires $dm/dv \geq 0$

Models of global quenches in holography: AdS-RN-Vaidya

- A further generalization of the AdS-Vaidya geometry is:

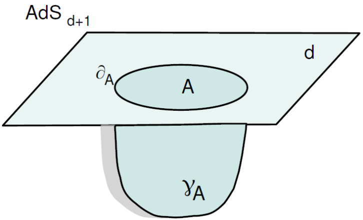
$$f(z,v) = 1 - z^d m(v) + q(v)^2 z^{2(d-1)}$$

which requires:

$$T_{\mu\nu} = \left(\frac{d-1}{4\kappa^2} z^{d-1} \frac{dm}{dv} - \frac{d-2}{2\kappa^2} z^{2d-3} q(v) \frac{dq}{dv}\right) \delta_\mu^v \delta_\nu^v$$

- Assuming $m(-\infty) = 0$, $q(-\infty) = 0$, it interpolates between AdS and AdS-RN (CFT vacuum to a state with finite T and μ).
- NEC is naively violated! but $m(v)$ and $q(v)$ are further constrained by SSA [Caceres, Kundu, Pedraza & Tangarife]
- Charge can be included in the perturbative collapse framework [Caceres, Kundu, Pedraza & Yang]

Holographic entanglement entropy

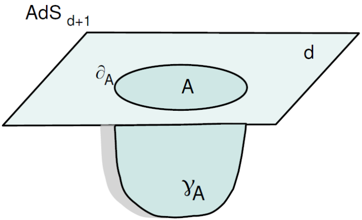


Prescription: [Ryu & Takayanagi]

$$S_A = \frac{\text{Area}_{\text{min}}(\gamma_A)}{4 G_N^{d+1}}$$

- γ_A = codimension-2 surface s.t. $\partial\gamma_A = \partial A$
- Homology constraint: $\gamma_A \sim A$
  - $\exists$ bulk region $\mathfrak{R}$ s.t. $\partial\mathfrak{R} = \gamma_A \cup A$

Holographic entanglement entropy



Covariant prescription: [Hubeny, Rangamani & Takayanagi]

$$S_A = \frac{\text{Area}_{\text{ext}}(\gamma_A)}{4G_N^{d+1}}$$

- γ_A = codimension-2 surface s.t. $\partial\gamma_A = \partial A$
- Homology constraint: $\gamma_A \sim A$
  - $\exists$ bulk region $\mathfrak{R}$ s.t. $\partial\mathfrak{R} = \gamma_A \cup A$

Spread of entanglement in $(1+1)$ dimensions

For $(1+1)$ CFTs EE is known in a closed form! [Balasubramanian et.al]
$\rightarrow$ Consider a segment of length $\ell = 2R$ and define:

$$\mathfrak{t} = 2\pi T t \,, \qquad \qquad \mathfrak{l} = 2\pi T R$$

- At $\mathfrak{t} \to \infty$ EE reaches the equilibrium value:

$$S_A = \frac{c}{3} \log\left(\frac{R}{\epsilon}\right) + \frac{c}{3} \log\left(\frac{\sinh \mathfrak{l}}{\mathfrak{l}}\right) \equiv S_{\text{vac}} + \Delta S_A$$

  - For $\mathfrak{l} \gg 1$: $\quad \Delta S_A \simeq \frac{c\mathfrak{l}}{3} = s_{\text{eq}} V_A \,, \qquad s_{\text{eq}} = \frac{\pi c T}{3}$

    The first law reads: $\quad \boxed{\left.\frac{d(\Delta E_A)}{d(\Delta S_A)}\right|_{\ell} = T}$

    where $\quad \Delta E_A = \mathcal{E} V_A \,, \qquad \mathcal{E} = \frac{\pi c T^2}{6}$

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  - For $\mathfrak{l} \ll 1$: $\qquad \Delta S_A \simeq \frac{c\mathfrak{l}^2}{18} = \frac{c\pi^2 T^2 \ell^2}{18}$

    The first law reads: $\quad \boxed{\left.\frac{d(\Delta E_A)}{d(\Delta S_A)}\right|_{\ell} = T_A}$ [Bhattacharya et.al]

    where $\qquad T_A = \frac{3}{\pi \ell} \qquad \rightarrow \qquad \Delta S_A = \frac{\Delta E_A}{T_A} = \frac{\mathcal{E} V_A}{T_A} = s_{\rm eq} V_A$

Spread of entanglement in $(1+1)$ dimensions

For $\mathfrak{t} \leq \mathfrak{t}_{\text{sat}} = \mathfrak{l}$ [Balasubramanian et.al]:

$$\boxed{S_A(\mathfrak{t}) = S_{\text{vac}} + \Delta S_A(\mathfrak{t})\,, \qquad \Delta S_A(\mathfrak{t}) = \frac{c}{3}\log\left(\frac{\sinh \mathfrak{t}}{\mathfrak{l}\, s(\mathfrak{l},\mathfrak{t})}\right)}$$

where

$$\mathfrak{l} = \frac{\sqrt{1-s^2}}{\rho s} + \frac{1}{2}\log\left(\frac{2(1+\sqrt{1-s^2})\rho^2 + 2s\rho - \sqrt{1-s^2}}{2(1+\sqrt{1-s^2})\rho^2 - 2s\rho - \sqrt{1-s^2}}\right)$$

$$\rho = \frac{1}{2}\coth \mathfrak{t} + \frac{1}{2}\sqrt{\frac{1}{\sinh^2 \mathfrak{t}} + \frac{1-\sqrt{1-s^2}}{1+\sqrt{1-s^2}}}$$

One key observation:

$$\boxed{v_E^{\text{avg}} = \langle \mathfrak{R}(t)\rangle = \frac{1}{s_{\text{eq}} A_\Sigma}\frac{\Delta S_A}{\Delta t} = \frac{1}{s_{\text{eq}} A_\Sigma}\frac{s_{\text{eq}} V_A}{t_{\text{sat}}} = \frac{R}{t_{\text{sat}}} = 1}$$

Spread of entanglement in $(1+1)$ dimensions

Two possibilities:

- $\mathfrak{R}(t) = 1 \;\; (\forall\, \mathfrak{l}, \mathfrak{t}) \to$ Causality constraint is OK
- $\max[\mathfrak{R}(t)] > 1 \to$ Causality constraint is violated

It suffices to analyze the early time behavior $\mathfrak{t} \ll \mathfrak{t}_{\text{sat}}$:

$$\rho = \frac{1}{\mathfrak{t}} + \frac{\mathfrak{t}}{12} + \cdots \,, \qquad s = \frac{\mathfrak{t}}{\mathfrak{l}}\left(\frac{1}{\mathfrak{t}} - \frac{\mathfrak{t}}{12} + \cdots\right)$$

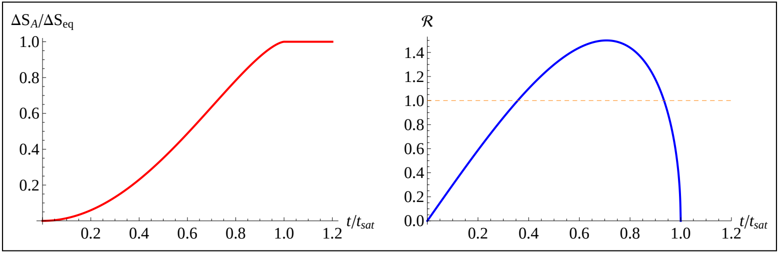
$$\Delta S_A(\mathfrak{t}) = \frac{c\mathfrak{t}^2}{12} + \mathcal{O}(\mathfrak{t}^4) = 2\pi\mathcal{E}t^2 + \cdots$$

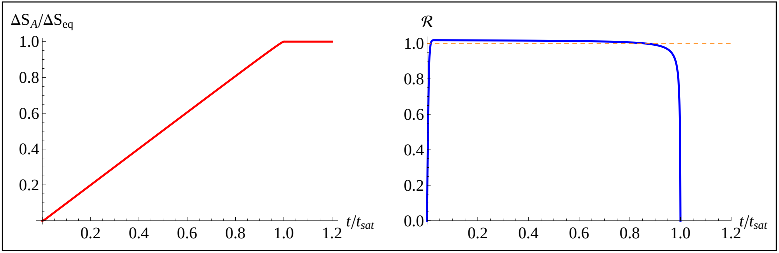
Therefore,

$$\boxed{\mathfrak{R}(t) = \frac{2\pi\mathcal{E}t}{s_{\text{eq}}} + \cdots} \qquad \to \qquad \max[\mathfrak{R}(t)] > 1!$$

Spread of entanglement in $(1+1)$ dimensions

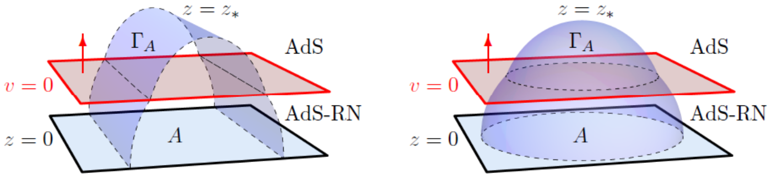
Numerical results for $\mathfrak{l} = 10^{-2}$ and $\mathfrak{l} = 10^{2}$. $\mathfrak{R}(t) \to 1$ as $\mathfrak{l} \to \infty$.





Perturbative calculation for small intervals

Setup in the gravity side:



- For large subsystems one solves for Γ_A above and below the shell and then match the solutions in a perturbatively in the IR [Liu & Suh]
- We developed a different perturbative expansion for small subsystems. In this case the extremal surfaces stay mostly in the UV [Kundu & Pedraza]

Perturbative calculation for small intervals

Let us expand the area functional $\mathcal{A}$ and embedding $\phi = \{x(z), v(z)\}$ as

$$\mathcal{A}[\phi(z);\lambda] = \mathcal{A}^{(0)}[\phi(z)] + \lambda\mathcal{A}^{(1)}[\phi(z)] + \mathcal{O}(\lambda^2)$$
$$\phi(z) = \phi^{(0)}(z) + \lambda\phi^{(1)}(z) + \mathcal{O}(\lambda^2)$$

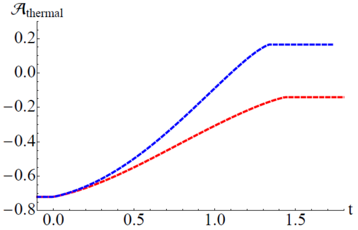
The key observation is that:

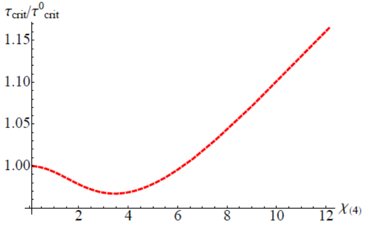
$$\mathcal{A}_{\text{on-shell}}[\phi(z)] = \int dz\, \mathcal{A}^{(0)}[\phi^{(0)}(z)] + \lambda \int dz\, \mathcal{A}^{(1)}[\phi^{(0)}(z)] + \lambda \int dz\, \phi_i^{(1)}(z) \left[\frac{d}{dz}\frac{\partial \mathcal{A}^{(0)}}{\partial \phi_i'(z)} - \frac{\partial \mathcal{A}^{(0)}}{\partial \phi_i(z)}\right]_{\phi^{(0)}} + \cdots$$

- We consider $\ell T \ll 1$, where $T \sim 1/z_H$
- Through the UV/IR connection $\ell \sim z$, so this corresponds to $z \ll z_H$ i.e. near the AdS boundary

Perturbative calculation for small intervals

Previous numerical results for AdS-RN-Vaidya [Caceres, Kundu]:





- Initial quadratic growth
- (Quasi)-linear intermediate regime
- Continuous saturation
- Non-monotonicity in the saturation time as a function of $\chi = \mu/T$

Perturbative calculation for small intervals: the strip

At first order we only need the embedding in pure AdS:

$$\Delta S(t) = S(t) - S_{\rm AdS} = \frac{\ell_\perp^{d-2}\varepsilon}{4G_N z_H^d}\int_0^{z_*} dz\,\theta(t-z)z\sqrt{1-(z/z_*)^{2(d-1)}}$$

which leads to

$$\Delta S(t) = \begin{cases} 0\,, & t<0 \\ \Delta S_{\rm eq}\times\mathcal{F}(t/t_{\rm sat})\,, & 0\le t\le t_{\rm sat}\,, \\ \Delta S_{\rm eq} & t>t_{\rm sat} \end{cases}$$

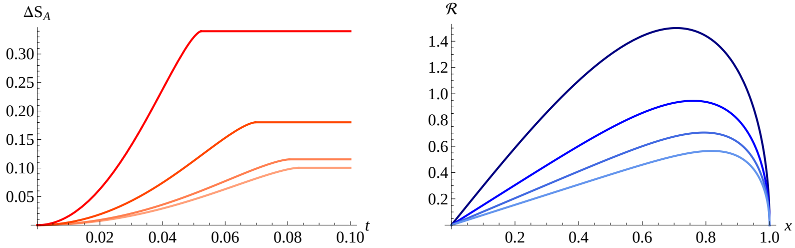
where

$$\Delta S_{\rm eq} = \frac{\sqrt{\pi}\Gamma[\frac{1}{d-1}]\ell_\perp^{d-2}\varepsilon z_*^2}{8(d+1)\Gamma[\frac{d+1}{2(d-1)}]z_H^d G_N^{(d+1)}}$$

$$\mathcal{F}(x) = \frac{2\Gamma[\frac{d+1}{2(d-1)}]x^2}{\sqrt{\pi}\Gamma[\frac{1}{d-1}]}\left[\sqrt{1-x^{2(d-1)}}+\tfrac{d-1}{2}\,{}_2F_1\left(\tfrac{1}{2},\tfrac{1}{d-1},\tfrac{d}{d-1},x^{2(d-1)}\right)\right]$$

Perturbative calculation for small intervals: the strip

Plots for $\Delta S(t)$ (for different μ/T) and $\mathfrak{R}(t)$ (for different d):



- Initial quadratic growth regime:

$$\Delta S = \begin{cases} \frac{A_\Sigma}{4G_N}\left(\frac{4\pi T}{d}\right)^d\left(1+\frac{d^2(d-2)}{16\pi^2}\left(\frac{\mu}{T}\right)^2+\cdots\right)t^2, & T \gg \mu \\ \frac{A_\Sigma}{4G_N}\frac{2(d-2)^{d-1}\mu^d}{d^{d/2}(d-1)^{d/2-1}}\left(1+\frac{2\pi d^{1/2}}{(d-1)^{1/2}}\frac{T}{\mu}+\cdots\right)t^2, & T \ll \mu \end{cases}$$

  - Same behavior than for large intervals [Liu & Suh]. Universal!

Perturbative calculation for small intervals: the strip

- Quasi-linear growth regime with:

$$\max[\mathfrak{R}(t)] = \frac{4(d-1)^{3/2}\Gamma[\frac{3d-1}{2(d-1)}]\Gamma[\frac{d}{2(d-1)}]}{d^{\frac{d}{2(d-1)}}\Gamma[\frac{1}{2(d-1)}]\Gamma[\frac{1}{d-1}]} = \begin{cases} 3/2\,, & d=2 \\ 0.946\,, & d=3 \\ 0.704\,, & d=4 \\ \pi/d \to 0\,, & d\to\infty \end{cases}$$

  - Causality constraint is violated only for $d = 2$.
  - However $\langle \mathfrak{R}(t) \rangle \leq 1$ $(\forall\ d)$.

- In this approximation $t_{\rm sat} = z_* \sim \ell$ and $t_{\rm sat}/t_{\rm sat}^{(0)} = 1$.
- At next order we find:

$$\frac{t_{\rm sat}}{t_{\rm sat}^{(0)}} = 1 - \alpha(d)\left(\frac{\mu}{T}\right)^2 (T\ell)^d + \mathcal{O}\left(\frac{\mu}{T}\right)^4 + \mathcal{O}\,(T\ell)^{2(d-1)} \qquad (2)$$

where $\alpha(d) > 0$! Next order is positive $\to$ non-monotonic.

Perturbative calculation for small intervals: the ball

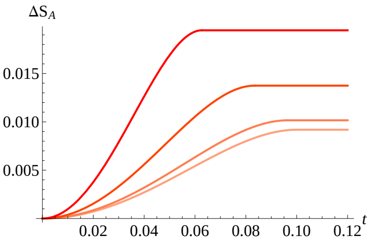
$$\Delta S(t) = \frac{\varepsilon A_\Sigma z_*^{d-2}}{8 G_N^{(d+1)} R^{d-2} z_H^d} \int_0^{z_*} dz\, \theta(t-z) z \left[1 - (z/z_*)^2\right]^{\frac{d-1}{2}}$$

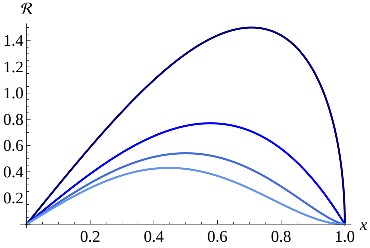
which leads to

$$\Delta S(t) = \begin{cases} 0\,, & t < 0 \\ \Delta S_{\text{eq}} \times \mathcal{G}(t/t_{\text{sat}})\,, & 0 \leq t \leq t_{\text{sat}} \\ \Delta S_{\text{eq}} & t > t_{\text{sat}} \end{cases}$$

where

$$\Delta S_{\text{eq}} = \frac{R^2 A_\Sigma \varepsilon}{8(d+1) z_H^d G_N^{(d+1)}}\,, \qquad \mathcal{G}(x) = 1 - \left(1 - x^2\right)^{\frac{d+1}{2}}$$





Linear response of entanglement entropy

- The entanglement growth looks like a convolution integral!

$$\Delta S_{\rm s}(t) \sim \int_0^{t_*} dt'\,\theta(t-t')t'\sqrt{1-(t'/t_*)^{2(d-1)}} = \mathfrak{f}(t) * \mathfrak{g}_{\rm s}(t)$$

$$\Delta S_{\rm b}(t) \sim \int_0^{t_*} dt'\,\theta(t-t')t'\left[1-\left(t'/t_*\right)^2\right]^{\frac{d-1}{2}} = \mathfrak{f}(t) * \mathfrak{g}_{\rm b}(t)$$

where $\mathfrak{f}(t) \sim \theta(t)$ is the source and

$$\mathfrak{g}_{\rm s}(t) \sim t\sqrt{1-(t/t_*)^{2(d-1)}}, \qquad \mathfrak{g}_{\rm b}(t) \sim t\left[1-\left(t/t_*\right)^2\right]^{\frac{d-1}{2}}$$

are response functions for the strip and the ball, respectively.

Q: Can we understand this statement better? $\mathfrak{f}(t)$ comes from the mass term $m(v)$, do these expressions hold for more general sources?

Fefferman-Graham expansion for AdS spaces

Any asymptotically AdS metric can be written as:

$$ds^2 = \frac{1}{\rho^2}\left(g_{\mu\nu}(\rho, x^\mu)dx^\mu dx^\nu + d\rho^2\right)$$

where $g_{\mu\nu}^{\text{CFT}} = g_{\mu\nu}(0, x^\mu)$. Assuming $g_{\mu\nu}(\rho, x^\mu) = \eta_{\mu\nu} + \delta g_{\mu\nu}(\rho, x^\mu)$:

$$\delta g_{\mu\nu} = a\,\rho^d\,\langle T_{\mu\nu}\rangle + \rho^{2d}\left(a_1\,\langle T_{\mu\alpha}\,T^{\alpha}{}_{\nu}\rangle + a_2\,\eta_{\mu\nu}\langle T_{\alpha\beta}\,T^{\alpha\beta}\rangle\right) + \ldots$$

Corrections from additional operators (dual to vector A_μ or scalar ϕ):

$$\delta g_{\mu\nu} = a\,\rho^d\,\langle T_{\mu\nu}\rangle + \rho^{2d-2}\left(b_1\,\langle J_\mu J_\nu\rangle + b_2\,\eta_{\mu\nu}\langle J_\alpha J^\alpha\rangle\right) + \ldots$$
$$\delta g_{\mu\nu} = a\,\rho^d\,\langle T_{\mu\nu}\rangle + c\,\rho^{2\Delta}\langle\mathcal{O}^2\rangle + \ldots$$

The last term can dominate if $\frac{d}{2} - 1 < \Delta < \frac{d}{2}$ (do not consider these)

Entanglement entropy for small subsystems in equilibrium

Indeed, for static spacetimes: [Bhattacharya et.al.]

$$\Delta S_A = \langle T_{00} \rangle \frac{V_A}{T_A} = \frac{\Delta E_A}{T_A}$$

where

$$T_{\rm s} = \frac{2(d^2-1)\Gamma[\frac{d+1}{2(d-1)}]\Gamma[\frac{d}{2(d-1)}]^2}{\sqrt{\pi}\Gamma[\frac{1}{d-1}]\Gamma[\frac{1}{2(d-1)}]^2\ell}\,, \qquad T_{\rm b} = \frac{d+1}{2\pi R}$$

Notice that T_A can also be obtained from the modular Hamiltonian:

$$H_{\rm b} = 2\pi \int_A \frac{R^2-r^2}{2R}\, T_{00}(x)\, d^{d-1}x$$

$$\downarrow$$

$$\Delta S_{\rm b} = 2\pi\delta\langle T_{00}(x)\rangle \Omega_{d-2} \int_0^R \frac{R^2-r^2}{2R} r^{d-2} dr = \frac{2\pi\delta\langle T_{00}(x)\rangle \Omega_{d-2} R^d}{d^2-1}$$

Holds for local quenches! [Nozaki, Numasawa & Takayanagi]

Entanglement entropy after general global quenches

For a general AdS-Vaidya:

$$\langle T_{00}(t)\rangle \equiv \varepsilon(t) = \frac{(d-1)m(t)}{16\pi G_N^{(d+1)}}\,, \quad \langle T_{ii}(t)\rangle \equiv P(t) = \frac{m(t)}{16\pi G_N^{(d+1)}}$$

And entanglement entropy satisfies:

$$\boxed{\Delta S_A(t) = \int_{-\infty}^{\infty} dt'\, \mathfrak{f}(t-t')\mathfrak{g}_A(t')}$$

where $\mathfrak{f}(t) = \varepsilon(t)$ and

$$\mathfrak{g}_{\rm s}(t) = \frac{2\pi A_\Sigma t}{d-1}\sqrt{1-(t/t_*)^{2(d-1)}}\,[\theta(t)-\theta(t-t_*)]$$

$$\mathfrak{g}_{\rm b}(t) = \frac{2\pi A_\Sigma t}{d-1}\left[1-(t/t_*)^2\right]^{\frac{d-1}{2}}[\theta(t)-\theta(t-t_*)]$$

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Can be integrated by parts to obtain:

$$\boxed{\Delta S_A(t) = \frac{\Delta E_A(t)}{T_A} + \int_{\infty}^{\infty} dt' \, \frac{d\varepsilon(t-t')}{dt} \mathfrak{G}_A(t')}$$

- First law recovered for adiabatic quenches, provided:

$$\frac{d\varepsilon(t')}{dt'} t_* \ll \varepsilon(t) \,, \qquad \forall\, t' \in (t - t_*, t)$$

- Second term can be interpreted as a kind of "relative entropy"

$$\Upsilon_A(t) = \frac{\Delta E_A(t)}{T_A} - \Delta S_A(t) \geq 0$$

  - Measures "distance" of out-of-equilibrium state w.r.t. equilibrium
  - Positivity implies SSA!
  - Can be rewritten using Ward identity, e.g. $\frac{d\varepsilon(t)}{dt} = \langle \mathcal{O}_\Delta(t) \rangle \frac{d\mathcal{J}_\Delta(t)}{dt}$

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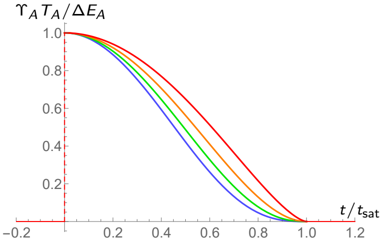
- Example 1. Instantaneous quench: $\varepsilon(t) = \varepsilon_0 \, \theta(t)$



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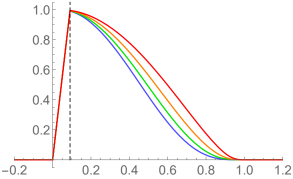
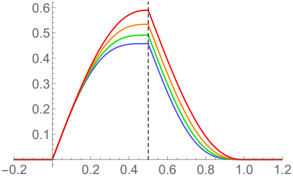
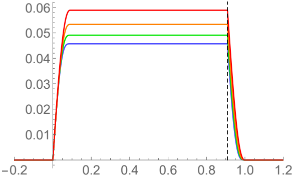
- Example 2. Linear-quench: $\varepsilon(t) = \alpha t\,[\theta(t) - \theta(t - t_q)] + \alpha t_q \theta(t - t_q)$







In the fully driven regime $\Upsilon_A =$ constant, so we recover [O'Bannon et.al.]

$$\boxed{\frac{d\Delta S_A(t)}{dt} = \frac{1}{T_A}\frac{d\Delta E_A(t)}{dt}}$$

Conclusions

- Spread of EE after instantaneous quenches follow universal rules:
  - For large subsystems the linear growth regime is universal, independent of the shape of Σ, but details depend on the final state (T, μ)
  - For small subsystems the evolution depends on Σ, but is universal with respect to the state (only depends on $\varepsilon(T, \mu)$)
- Causality constrains the instantaneous rate of EE growth only for large subsystems. Average growth is always bounded by the speed of light.
- Spread of EE for small systems is described by a linear response
  - Q: Field theory interpretation of $\mathfrak{g}_A(t)$?
  - Q: Fluctuation-dissipation theorems?
- The quantity $\Upsilon_A(t)$ resembles a "relative entropy". Useful order parameter to characterize out-of-equilibrium excited states

Conclusions

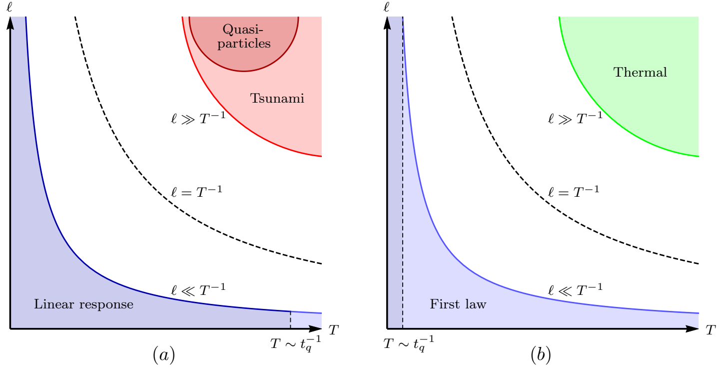


Figure: Schematic diagram of the different regimes of interest of entanglement propagation for (a) fast quenches $t_q \to 0$ and (b) slow quenches $t_q \to \infty$.