Sum rules for leading vector form factors in hyperon semileptonic decays

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- By considering that the weak and electromagnetic currents are members of the same $SU(3)$ octet,two sum rules involving leading vector form factors in hyperon semileptonic decays are derived in the limit of exact flavor $SU(3)$ symmetry[3].
- According to the Ademollo-Gatto theorem, violations to this sum rules are present at second-order $SU(3)$ symmetry-breaking.
- One of the sum rules does not acquire any violations in the presence of second-order $SU(3)$ symmetry-breaking and the other one obtains contributions form the $\mathbf{10} + \overline{\mathbf{10}}$ representation.

- Motivated by the success of the GMO mass formula for baryon octet masses,T. N. Pham observed that the matrix elements of the $V = 1$ V-spin multiplet can be related to each other in the exact flavor $SU(3)$ symmetry limit[6]. Starting from the following two I-spin relations:

$$\langle I = 1, I_3 = 0|\, \bar{u}\Gamma^n d\, |I = 1, I_3 = 1\rangle = - \langle I = 1, I_3 = -1|\, \bar{u}\Gamma^n d\, |I = 1, I_3 = 0\rangle \quad (1)$$

$$\langle I = 0, I_3 = 0|\, \bar{u}\Gamma^n d\, |I = 1, I_3 = 1\rangle = \langle I = 1, I_3 = -1|\, \bar{u}\Gamma^n d\, |I = 0, I_3 = 0\rangle \quad (2)$$

- The rotated V-spin versions of (1) and (2) read:

$$\langle V=1, V_3=0|\, \bar{u}\Gamma^n s \,|V=1, V_3=1\rangle = \\ -\langle V=1, V_3=-1|\, \bar{u}\Gamma^n s \,|V=1, V_3=0\rangle \quad (3)$$

$$\langle V=0, V_3=0|\, \bar{u}\Gamma^n s \,|V=1, V_3=1\rangle = \\ \langle V=1, V_3=-1|\, \bar{u}\Gamma^n s \,|V=0, V_3=0\rangle \quad (4)$$

- Decomposing the $SU(3)$ octet in eigenfunctions of V^2 yields:

$$|\Xi^-\rangle = |V = 1, V_3 = 1\rangle \tag{5}$$

$$\left|\frac{1}{2}\Sigma^0 + \frac{\sqrt{3}}{2}\Lambda\right\rangle = |V = 1, V_3 = 0\rangle \tag{6}$$

$$|p\rangle = |V = 1, V_3 = -1\rangle \tag{7}$$

$$\left|\frac{\sqrt{3}}{2}\Sigma^0 - \frac{1}{2}\Lambda\right\rangle = |V = 0, V_3 = 0\rangle \tag{8}$$

- Relations (3) and (4) apply to matrix elements of any $SU(3)$ octet $\Delta S = 1$ operator.
- GMO mass formula is trivially obtainded by setting $\Gamma^n = \mathbb{I}$ and relating $\langle B_2| \bar{u}s |B_1\rangle$ to $\left[f_1^{SU(3)}\right]_{B_1 B_2} (M_{B_2} - M_{B_1})$ in (3).
- When $\Gamma^n = \gamma_5\gamma_\mu$ some interesting relations for the axial-vector to vector form factor ratios $\frac{g_1}{f_1}$ can also be found.
- This analysis can be extended to the vector current by using $\Gamma^n = \gamma_\mu$.

- As a result of the above mentioned, two simple although nontrivial expressions are obtained[3]:

$$\left[f_1^{SU(3)}\right]_{\Xi^-\Sigma^0} + \sqrt{3}\left[f_1^{SU(3)}\right]_{\Xi^-\Lambda} + \frac{1}{\sqrt{2}}\left[f_1^{SU(3)}\right]_{\Sigma^-n} + \sqrt{3}\left[f_1^{SU(3)}\right]_{\Lambda p} = 0 \quad (9)$$

$$\sqrt{3}\left[f_1^{SU(3)}\right]_{\Xi^-\Sigma^0} - \left[f_1^{SU(3)}\right]_{\Xi^-\Lambda} - \sqrt{\frac{3}{2}}\left[f_1^{SU(3)}\right]_{\Sigma^-n} + \left[f_1^{SU(3)}\right]_{\Lambda p} = 0 \quad (10)$$

- Violations to (9) and (10) are expected to occur due to flavor $SU(3)$ symmetry-breaking.

- In the presence of symmetry-breaking, sum rules (9) and (10) can be expressed as:

$$\frac{1}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Sigma^0} + \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Lambda} - \frac{1}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^- n} - \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Lambda p} = \delta_1^{SB} \quad (11)$$

$$\frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Sigma^0} - \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Lambda} + \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^- n} - \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Lambda p} = \delta_2^{SB} \quad (12)$$

- δ^{SB} arises from flavor $SU(3)$ second-order symmetry breaking

- Assuming that the vector and electromagnetic currents are members of the same unitary octet and that the breaking of the unitary system behaves as the eight component of an octet, Ademollo and Gatto set up an important theorem on the nonrenormalization for the strangeness-violating vector currents.

- According to Ademollo and Gatto[1], to first-order in symmetry-breaking, the ath component of the vector current $\mathcal{J}^a$ can be written as:

$$
\begin{aligned}
\mathcal{J}^a + \epsilon\delta\mathcal{J}^a &= a_0 Tr\left(\bar{B}B\lambda^a\right) + b_0 Tr\left(\bar{B}\lambda^a B\right) \\
&+ \epsilon a\left[Tr\left(\bar{B}B\left\{\lambda^a,\lambda^8\right\}\right) - \frac{1}{8}\delta^{a8} Tr\left(\bar{B}B\left\{\lambda^c,\lambda^c\right\}\right)\right] \\
&+ \epsilon b\left[Tr\left(\bar{B}\left\{\lambda^a,\lambda^8\right\}B\right) - \frac{1}{8}\delta^{a8} Tr\left(\bar{B}\left\{\lambda^c,\lambda^c\right\}B\right)\right] \\
+ \epsilon c&\left[Tr\left(\bar{B}\lambda^a B\lambda^8\right) - Tr\left(\bar{B}\lambda^8 B\lambda^a\right)\right] + \epsilon g Tr\left(\bar{B}B\right) Tr\left(\lambda^a\lambda^8\right) \\
&+ \epsilon h[Tr\left(\bar{B}\lambda^a B\lambda^8\right) + Tr\left(\bar{B}\lambda^8 B\lambda^a\right) - \frac{1}{8}\delta^{a8} Tr\left(\bar{B}\lambda^c B\lambda^c\right) \\
&- \frac{6}{5} d^{a8c} d^{cgh} Tr\left(\bar{B}\lambda^g B\lambda^h\right)] \qquad (13)
\end{aligned}
$$

- B represents the baryon matrix and λ^a denotes the Gell-Mann matrices.
- a_0, b_0, a, b, c, g and h are coupling constants.
- Parameter ϵ is introduced to keep track of how many times symmetry-breaking to first-order is present; at the end of the calculation ϵ can be set to one without any loss of generality.

- The electromagnetic current is defined as:

$$\mathcal{J}_{em} = \mathcal{J}^{Q} \equiv \mathcal{J}^3 + \frac{1}{\sqrt{3}}\mathcal{J}^8 \tag{14}$$

- To first-order in symmetry-breaking, e.g. the electric charge for Λ reads:

$$Q_\Lambda + \epsilon\delta Q_\Lambda = -\frac{1}{3}a_0 - \frac{1}{3}b_0 + \frac{2}{\sqrt{3}}\epsilon a + \frac{2}{\sqrt{3}}\epsilon b + \frac{2}{\sqrt{3}}\epsilon g + \frac{9\sqrt{3}}{10}\epsilon h \tag{15}$$

- Similar expressions are obtained for the remaining baryon octet members.

- The system of linear equations for the coupling constants a_0, b_0, ..., h can be solved by using the important property that the electric charge remains unrenormalized to all orders in perturbation theory.
- Solving for a_0, b_0, ..., h yields:

$$a_0 = -\frac{1}{2}, b_0 = \frac{1}{2}, a = b = c = g = h = 0 \tag{16}$$

- The above result shows that the electromagnetic and vector current are unrenormalized to first-order in symmetry-breaking.
- This is in essence the celebrated result discovered by Ademmollo and Gatto.

- In the large N_c limit, the baryon sector has a contracted $SU(2) \otimes SU(N_f)$ spin-flavor symmetry, where N_f is the number of light quark flavors.
- For $N_f = 3$, the lowest-lying baryon states fall into a representation of the spin-flavor group $SU(6)$.
- The $1/N_c$ expansion of any QCD operator transforming according to a given $SU(2) \otimes SU(N_f)$ representation can be written in terms of n-body operators $\mathcal{O}_n$ as[2]:

$$\mathcal{O}_{QCD} = \sum_n c_{(n)} \frac{1}{N_c} \mathcal{O}_n \tag{17}$$

- Coefficients $c_{(n)}$ have power series expansions in $1/N_c$ beginning at order unity.
- Operators $\mathcal{O}_n$ are polynomials in the spin-flavor generators J^k, T^c and G^{kc}, which can be written as 1-body quark operators:

$$J^k = \sum_{\alpha}^{N_c} \bar{q}_\alpha \left(\frac{\sigma^k}{2} \otimes \mathbb{I} \right) q_\alpha \tag{18}$$

$$T^c = \sum_{\alpha}^{N_c} \bar{q}_\alpha \left(\mathbb{I} \otimes \frac{\lambda^c}{2} \right) q_\alpha \tag{19}$$

$$G^{kc} = \sum_{\alpha}^{N_c} \bar{q}_\alpha \left(\frac{\sigma^k}{2} \otimes \frac{\lambda^c}{2} \right) q_\alpha \tag{20}$$

- Where $\bar{q}_\alpha$ and q_α are operators that create and annihilate states in the fundamental representation of $SU(6)$.

- On general grounds, flavor symmetry-breaking in QCD is due to the strange quark mass m_s and transforms as a flavor octet.
- To first-order in symmetry-breaking, corrections are obtained from the tensor product $(\mathbf{0},\mathbf{8})\otimes(\mathbf{0},\mathbf{8})$ so that the $SU(2)\otimes SU(3)$ representations involved are $(\mathbf{0},\mathbf{1})$, $(\mathbf{0},\mathbf{8})$, $(\mathbf{0},\mathbf{10}+\overline{\mathbf{10}})$ and $(\mathbf{0},\mathbf{27})$.

- Let $V^c + \epsilon\delta V^c$ be the vector current containing the most general first-order symmetry-breaking terms in the $1/N_c$ expansion formalism[3]:

$$\begin{aligned}V^c+\epsilon\delta V^c &= c_{(1)}^{\mathbf{8}} T^c + c_{(2)}^{\mathbf{8}} \frac{1}{N_c}\{J^r, G^{rc}\} + \epsilon N_c a_{(0)}^{\mathbf{1}} + \epsilon N_c a_{(1)}^{\mathbf{8}} d^{c8e} T^e \\ &+ \epsilon a_{(2)}^{\mathbf{8}} \frac{1}{N_c} d^{c8e}\{J^r, G^{re}\} + \epsilon a_{(2)}^{\mathbf{27}} \frac{1}{N_c}\{T^c, T^8\} \\ &+ \epsilon a_{(3)}^{\mathbf{10}+\overline{\mathbf{10}}} \frac{1}{N_c^2}\left(\{T^c,\{J^r, G^{r8}\}\} - \{T^8,\{J^r, G^{rc}\}\}\right) \\ &+ a_{(3)}^{\mathbf{27}} \frac{1}{N_c^2}\left(\{T^c,\{J^r, G^{r8}\}\} + \{T^8,\{J^r, G^{rc}\}\}\right) \qquad (21)\end{aligned}$$

- It is important to stress the following points:
  1. $(\mathbf{0},\mathbf{1})$ and $(\mathbf{0},\mathbf{8})$ representations have to be subtracted from $(\mathbf{0},\mathbf{10}+\overline{\mathbf{10}})$ and $(\mathbf{0},\mathbf{27})$ representations.
  2. Since $N_c = 3$ up to three-body operators are retained.
- When $c = 3 + \frac{1}{\sqrt{3}}8$, the matrix elements of the operator $V^c + \epsilon\delta V^c$ between $SU(6)$ symmetric states give the actual values of baryon electric charges Q_B.

- For the physical values $N_c = N_f = 3$, the electric charge for Λ reads:

$$Q_\Lambda + \epsilon\delta Q_\Lambda = -\frac{1}{6}c^{\mathbf{8}}_{(2)} + \sqrt{3}\epsilon a^{\mathbf{1}}_{(0)} + \frac{1}{6\sqrt{3}}\epsilon a^{\mathbf{8}}_{(2)} - \frac{3\sqrt{3}}{20}x^{\mathbf{27}} \tag{22}$$

- Where $x^{\mathbf{27}} \equiv a^{\mathbf{27}}_{(2)} + \frac{2}{3}a^{\mathbf{27}}_{(3)}$.
- Similar expressions are obtained for the remaining baryon octet members.

- In order to prove that the Ademollo-Gatto theorem in the $1/N_c$ expansion formalism it is necessary to solve the linear system of equations involving $c_{(n)}^{\mathbf{rep}}$ and $a_{(n)}^{\mathbf{rep}}$.
- It is important to take into account the following points:
  1. Electric charge remains unrenormalized to all orders in perturbation theory.
  2. The number of unkwnons is reduced by one since $x^{\mathbf{27}}$ appears in all the expressions for the electric charge of the baryon octet.
- As expected, the solution for the above system yields:

$$c_{(1)}^{\mathbf{8}} = 1, c_{(2)}^{\mathbf{8}} = 0, a_{(0)}^{\mathbf{1}} = 0, a_{(1)}^{\mathbf{8}} = 0, a_{(2)}^{\mathbf{8}} = 0, x^{\mathbf{27}} = 0, a_{(3)}^{\mathbf{10}+\bar{\mathbf{10}}} = 0 \tag{23}$$

- The above nicely reproduces the Ademollo-Gatto result.

- Since the number of unknowns (coupling constants) in the original derivation of Ademollo-Gatto and the $1/N_c$ formalism are the same it is possible to find a *one-to-one* relation between the coupling constants of the Ademollo-Gatto and the $1/N_c$ formalism given by:

$$\begin{pmatrix} a_0 \\ b_0 \\ \vdots \\ h \end{pmatrix} = \mathbf{A_1}^{-1}\mathbf{A_2} \begin{pmatrix} c^{\mathbf{8}}_{(1)} \\ c^{\mathbf{8}}_{(2)} \\ \vdots \\ x^{\mathbf{27}} \end{pmatrix} \tag{24}$$

- $\mathbf{A_1}$ and $\mathbf{A_2}$ are 7×7 matrices that have the numeric coefficients that multiply the coupling constants of the Ademollo-Gatto and $1/N_c$ expansion respectively.

$$
\begin{aligned}
\epsilon^2 \delta V^c = {} & \epsilon^2 b_{(0)}^{\mathbf{1}} N_c d^{c88} + \epsilon^2 b_{(1)}^{\mathbf{8}} \delta^{c8} T^8 + \epsilon^2 e_{(1)}^{\mathbf{8}} f^{c8e} f^{8eg} T^g \\
& + \epsilon^2 g_{(1)}^{\mathbf{8}} d^{c8e} d^{8eg} T^g + \epsilon^2 h_{(1)}^{\mathbf{8}} \left(\imath f^{ceg} d^{8e8} T^g - \imath d^{ce8} f^{8eg} T^g - \imath f^{c8e} d^{eg8} T^g \right) \\
& + \epsilon^2 b_{(2)}^{\mathbf{8}} \frac{1}{N_c} \delta^{c8} \left\{ J^r, G^{r8} \right\} + \epsilon^2 e_{(2)}^{\mathbf{8}} f^{c8e} f^{8eg} \frac{1}{N_c} \left\{ J^r, G^{rg} \right\} \\
& + \epsilon^2 g_{(2)}^{\mathbf{8}} d^{c8e} d^{8eg} \frac{1}{N_c} \left\{ J^r, G^{rg} \right\} \\
& + \epsilon^2 h_{(2)}^{\mathbf{8}} \left(\imath f^{ceg} d^{8e8} - \imath d^{ce8} f^{8eg} - \imath f^{c8e} d^{eg8} \right) \left\{ J^r, G^{rg} \right\} \\
& + \epsilon^2 b_{(3)}^{\mathbf{10}+\overline{\mathbf{10}}} \frac{1}{N_c^2} d^{c8e} \left(\left\{ T^e, \left\{ J^r, G^{r8} \right\} \right\} - \left\{ T^8, \left\{ J^r, G^{re} \right\} \right\} \right) \\
& + \epsilon^2 b_{(2)}^{\mathbf{27}} \frac{1}{N_c} d^{c8e} \left\{ T^e, T^8 \right\} + \epsilon^2 b_{(3)}^{\mathbf{64}} \frac{1}{N_c^2} \left\{ T^c, \left\{ T^8, T^8 \right\} \right\} \\
& + \epsilon^2 b_{(3)}^{\mathbf{27}} \frac{1}{N_c^2} d^{c8e} \left(\left\{ T^e, \left\{ J^r, G^{r8} \right\} \right\} + \left\{ T^8, \left\{ J^r, G^{re} \right\} \right\} \right)
\end{aligned} \tag{25}
$$

- Matrix elements of the operator $\epsilon^2\delta V^Q$ between $SU(6)$ symmetric states gives the second-order symmetry-breaking effects for the baryon octet electric charges. For neutron, these corrections read:

$$\epsilon^2\delta Q_n = -\epsilon^2 b^{\mathbf{1}}_{(0)} + \frac{1}{2}\epsilon^2 b^{\mathbf{8}}_{(1)} + \frac{1}{12}\epsilon^2 b^{\mathbf{8}}_{(2)} + \frac{1}{3}\epsilon^2 g^{\mathbf{8}}_{(1)} + \frac{1}{6}\epsilon^2 g^{\mathbf{8}}_{(2)} - \frac{1}{9}\epsilon^2 b^{\mathbf{10}+\overline{\mathbf{10}}}_{(3)} - \frac{1}{15}\epsilon^2 b^{\mathbf{27}}_{(2)} - \frac{2}{45}\epsilon^2 b^{\mathbf{27}}_{(3)} + \frac{13}{120}\epsilon^2 b^{\mathbf{64}}_{(3)} \quad (26)$$

- Similar expressions are found for the rest of the baryon charges.

- Again, the operator coefficients associated with the **27** representation can be absorbed in a new coefficient $y^{\mathbf{27}} = b_{(2)}^{\mathbf{27}} + \frac{2}{3} b_{(3)}^{\mathbf{27}}$.
- By using the important property that the electric charge remains unrenormalized to all orders in perturbation theory, the system can be solved as follows:

$$b_{(0)}^{\mathbf{1}} = b_{(2)}^{\mathbf{8}} = g_{(2)}^{\mathbf{8}} = y^{\mathbf{27}} = 0 \tag{27}$$

$$g_{(1)}^{\mathbf{8}} = -\frac{5}{24} b_{(3)}^{\mathbf{10}+\overline{\mathbf{10}}} \tag{28}$$

$$b_{(1)}^{\mathbf{8}} = -\frac{13}{72} b_{(3)}^{\mathbf{10}+\overline{\mathbf{10}}} \tag{29}$$

$$b_{(3)}^{\mathbf{64}} = \frac{5}{2} b_{(3)}^{\mathbf{10}+\overline{\mathbf{10}}} \tag{30}$$

- Thus, under the working assumptions, the baryon vector current is given to second-order in flavor symmetry-breaking in terms of, in principle, five non-trivial operator coefficients.
- For example, leading vector form factor for $\Lambda \to p$ yields:

$$\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Lambda p} = 1 + \frac{3}{4}e_{(1)}^{\mathbf{8}} - \frac{1}{2}h_{(1)}^{\mathbf{8}} + \frac{3}{8}e_{(2)}^{\mathbf{8}} - \frac{1}{4}h_{(2)}^{\mathbf{8}} + \frac{11}{288}b_{(3)}^{\mathbf{10}+\bar{\mathbf{10}}} \quad (31)$$

- Similar expressions are obtained for the remaining transitions $\Sigma^- \to n$, $\Xi^- \to \Sigma^0$ and $\Xi^- \to \Lambda$

- Substituting the obtained expressions for leading vector form factors in (11) and (12) yields:

$$\frac{1}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Sigma^0} + \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Lambda} - \frac{1}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^- n} - \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Lambda p} = 0 = \delta_1^{SB} \quad (32)$$

$$\frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Sigma^0} - \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Xi^-\Lambda} + \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Sigma^- n} - \frac{3}{4}\left[\frac{f_1}{f_1^{SU(3)}}\right]_{\Lambda p} = -\frac{1}{24} b_{(3)}^{\mathbf{10}+\overline{\mathbf{10}}} = \delta_2^{SB} \quad (33)$$

- The above findings are remarkable, sum rule (32) is valid in the presence of second-order symmetry-breaking whereas sum rule (33) gets corrections from the $\mathbf{10}+\overline{\mathbf{10}}$ representation.
- Different methods have been used to evaluate symmetry-breaking effects to vector form factors. One of them is baryon chiral perturbation theory (BChPT) to order $\mathcal{O}(p^2)$ in the works by Krause[4].
- Later on, Villadoro[7] used heavy baryon chiral perturbation theory (HBChPT) with both octet and decuplet baryon degrees of freedom and included (partially) up order $\mathcal{O}(p^3)$ corrections corresponding to subleading in $1/M_B$ terms.
- In the context of covariant BChPT with the IR regularization, Lacour, Kubis and Meissner[5] performed calculations to order $\mathcal{O}(p^3)$ including only octet baryons as active degrees of freedom.

- Sum rule (32) is fulfilled by all the expressions for the $f_1/f_1^{SU(3)}$ ratios obtained within (heavy) baryon chiral perturbation theory to order $\mathcal{O}(p^2)$ of [4],[5] and [7].

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