Feynman's path integral and Henstock integral.

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Schrödinger equation

The Schrödinger equation describes the evolution of a state function for a particle of a constant mass m moving in Euclidean space $\mathbb{R}^d$ in the presence of a potential $V(x)$.

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= i[\frac{1}{2}\Delta - V]\psi \\ \psi|_{t=0} &= \varphi, \end{aligned}$$

here, $\hbar = 1$, Δ is the Laplacian operator in $\mathbb{R}^d$ and $V : \mathbb{R}^d \to \mathbb{R}$ is a measurable function.

Feynman's postulates

First postulate

A probability that the particle takes a path in a certain region R of space-time is given by a square of a complex number

$$|\varphi(R)|^2.$$

The number $\varphi(R)$ is called the probability amplitude for region R.
Feynman's first postulate gives a formula to compute it as **"a sum of complex contributions, one from each path in the region."**

$$\varphi(R) = \lim_{\epsilon \to 0} \int_R \Phi(\ldots x_i, x_{i+1}, \ldots)\ldots dx_i dx_{i+1} \ldots$$

Here $\Phi(\ldots xi, xi + 1\ldots)$ is some complex valued function.

Second postulate (formula to obtain Φ)

"The paths contribute equally in magnitude, but the phase of their contribution is the classical action (in units of $\hbar$); i.e. the time integral of the Lagrangian taken along the path."

The value of Φ for each path $x(t)$ is proportional to $e^{iS(x(t))}$,

$$\Phi[x(t)] \sim e^{iS(x(t))}$$

where $S(x(t))$ is the classical action, that is the time integral of the Lagrangian along the path of the particle

$$S(x(t)) = \int L(x'(t), x(t))dt$$

According to Hamilton's principle the classical path minimizes the action and second postulate we have: the probability amplitude is given by

$$\varphi(R) = \lim_{\epsilon \to 0} \int_R e^{i \sum S(x_{i+1}, x_i)} \ldots \frac{dx_i}{A} \frac{dx_{i+1}}{A} \ldots$$

According to Hamilton's principle the classical path minimizes the action and second postulate we have: the probability amplitude is given by

$$\begin{aligned} \varphi(R) &= \lim_{\epsilon\to 0}\int_R e^{i\sum S(x_{i+1},x_i)}\ldots\frac{dx_i}{A}\frac{dx_{i+1}}{A}\ldots \qquad (1)\\ (by\ \ PT) &= \lim_{n\to\infty}\left(\frac{mn}{2\pi it}\right)^{nd/2}\underbrace{\int_{\mathbb{R}^d}\ldots\int_{\mathbb{R}^d}}_{n\ \ veces} e^{i\sum[\frac{m}{2}\frac{(x_j-x_{j-1})^2}{(t/n)^2}-V(x_j)]^{t/n}}\varphi(x_n)dx_1\ldots dx_n \end{aligned}$$

where the factors $\frac{1}{A}$ are called the normalization factors.

Feynman's physical intuition

Feynman's fundamental postulate states that a resulting path of a particle is a sum over all paths and writes it in the form of an integral over the space of all paths.

$$Cost \int_{\Omega_x} e^{iS(w,t)} \varphi(w(t)) Dw \tag{2}$$

$$\psi = Cost \int_{\Omega_x} e^{iS(w,t)} \varphi(w(t)) Dw$$

$$= \lim_{n\to\infty} \left(\frac{mn}{2\pi it}\right)^{nd/2} \underbrace{\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d}}_{n\ veces} e^{i\sum[\frac{m}{2}\frac{(x_j - x_{j-1})^2}{(t/n)^2} - V(x_j)]^{t/n}} \varphi(x_n) dx_1 \ldots dx_n \qquad (3)$$

Difficulties

- The normalization constant "constant" has a meaning for every finite n, but it becomes infinite as n approaches infinity.

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- We need to interpret $S(x_0, \ldots, x_n; t)$ as a classical action. A path of a Brownian particle is continuous but with probability one nowhere differentiable function.

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- The normalization constant "constant" has a meaning for every finite n, but it becomes infinite as n approaches infinity.
- We need to interpret $S(x_0, ..., x_n; t)$ as a classical action. A path of a Brownian particle is continuous but with probability one nowhere differentiable function.
- In (3) $Dw = \lim_{n\to\infty} \prod_{j=1}^{n} dx_j$ corresponds to some measure on the space of all possible paths, but this product is infinite, thus defined this way, the measure Dw has no firm mathematical meaning.

General objective

Formalize the Feynman path integral using the Henstock integral.

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Particular objective

Present the main idea of the Henstock integral.

Probability distribution function

Let T be an interval in $(0, \infty)$.

$$\mathbb{R}^T = \{(x_t)_{t \in T}\},$$

the set of real-valued functions defined on T.
Let us denote $\mathcal{N} = N(T) = \{N = \{t_1, t_2, ..., t_n\} \subset T, n \in \mathbb{N}\}$.

- A figure in $\mathbb{R}^T$ is the union of a finite number of cells, denoted by E.
- A cell in $\mathbb{R}^T$ is

$$I[N] = I_1 \times I_2 \times ... \times I_n \times \mathbb{R}^{T \setminus N} = I(N) \times \mathbb{R}^{T \setminus N},$$

each I_i is an interval in $\overline{\mathbb{R}}$.

$g^T : \overline{\mathbb{R}}^T \times \mathcal{N} \to \mathbb{C}$ and $G^T : E(\mathbb{R}^T) \to \mathbb{C}$, and $E(\mathbb{R}^T)$ denotes the collection of all figures in $\mathbb{R}^n$.

$$g^T(x(N))|I[N]| = \begin{cases} \left(\sqrt{\frac{-i}{2\pi}}\right)^n e^{\frac{i}{2}x_1^2+\ldots+x_n^2}|I[N]| & \text{if} \quad x \in \mathbb{R}^T \\ \left(\sqrt{\frac{-i}{2\pi}}\right)^n \prod_{j=1}^n \int_{I_j} e^{\frac{i}{2}x_j^2} dx_j & \text{if} \quad x \in \overline{\mathbb{R}}^T \setminus \mathbb{R}^T \end{cases} . \tag{4}$$

$$G^T(I[N]) = \left(\sqrt{\frac{-i}{2\pi}}\right)^n \prod_{j=1}^n \int_{I_j} e^{\frac{i}{2}x_j^2} dx_j, \tag{5}$$

Then,

$$\psi = \int_{\mathbb{R}^T} e^{\frac{-i}{2}\int_T U(x(t),t)dt} g^T(x(N))|I[N]|,$$

where U is a real valued function.

[P. Muldowney, Theorem 154]

G^T is a probability distribution function. It menas, G^T is finitely additive on disjoint figures and

$$G^T(\mathbb{R}^T) = 1.$$

Main idea of Henstock integral in one dimension

Definition

A gauge function $\delta : [a, \infty] \to \mathbb{R}$ *is a positive function.*

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It is said that $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n+1}$ *is a tagged partition of* $[a, \infty]$ *if* $\{[x_{i-1}, x_i]\}_{i=1}^{n+1}$ *is a partition of* $[a, \infty]$ *and the points* $t_i \in [x_{i-1}, x_i]$ *are called tags.*

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Definition

A tagged partition $P = \{([x_0, x_1], t_1), ...([x_{n-1}, x_n], t_n), ([x_n, \infty], t_{n+1})\}$ *of* $[a, \infty]$ *is* $\delta -$ *fine if*

- $[x_{i-1}, x_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)],\ i = 1, ...n$
- $[x_n, \infty] \subset [1/\delta(\infty), \infty]$,

Definition

It is said that a function $f : [a, \infty] \to \mathbb{R}$ *is Henstock-Kurzweil integrable on* $[a, \infty]$*, if there exists* $A \in \mathbb{R}$ *such that, for each* $\varepsilon > 0$, *there is a gauge* $\gamma_\varepsilon : [a, \infty] \to (0, \infty)$ *such that if* $P = \{([x_0, x_1], t_1), ...([x_{n-1}, x_n], t_n), ([x_n, \infty], t_{n+1})\}$ *is a tagged partition* $\gamma_\varepsilon -$ *fine of* $[a, \infty]$ *then,*

$$\left| \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A \right| < \varepsilon.$$

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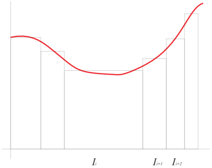
$$[-\infty, a] \quad [-\infty, \infty].$$



Definitions in $\mathbb{R}^n$

Definition

A cell I in $\mathbb{R}^n$ consists in the product $I = I(N) = I_1 \times ... \times I_n$, where $N = \{1, ..., n\}$ and each I_j can have the form

$$(-\infty, a), \quad [u, v], \quad (b, \infty], \quad \text{or} \quad (-\infty, \infty).$$

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Let I be a cell in $\mathbb{R}^n$. The cell is associated to $x \in \overline{\mathbb{R}}^n$ if for each $j = 1, ..., n$

- $x_j = -\infty$, if $I_j = (-\infty, a]$
- $x_j = u$ or $x_j = v$, if $I_j = [u, v]$
- $x_j = \infty$, if $I_j = (b, \infty)$
- $x_j = -\infty$ or $x_j = \infty$, if $I_j = (-\infty, \infty)$

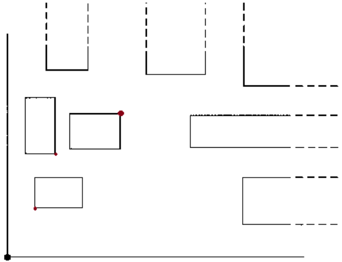

Figure : Some elements of a partition of $[0, \infty] \times [0, \infty]$.

Definition

A gauge in $\mathbb{R}^n$ *is a positive function* δ *defined for* $x \in \overline{\mathbb{R}}^n$. *An attached point-cell pair* (x, I) *of* $\mathbb{R}^n$ *is* $\delta-$ *fine if, for each* j, *the pair* (x_j, I_j) *is* $\delta-$ *fine in* $\mathbb{R}$; *that is,*

$$a < \frac{-1}{\delta(x)}; \quad u - v < \delta(x); \quad \text{or} \quad u > \frac{1}{\delta(x)}, \quad \text{respectively.}$$

- A **division** $\mathcal{D}$ of $\mathbb{R}^n$ is a finite collection of attached point-cell pairs (x, I) whose cells form a partition of $\mathbb{R}^n$.
- Given a gauge $\delta : \overline{\mathbb{R}}^n \to \mathbb{R}^+$, a **division** $\mathcal{D}$ **is** $\delta -$ *fine* if each $(x, I) \in \mathcal{D}$ is $\delta -$ *fine*, it is denoted by $\mathcal{D}_\delta$.

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Definition

A function $h(x, I)$ is integrable in $\mathbb{R}^n$ with integral $\alpha = \int_{\mathbb{R}^n} h(x, I)$ if, given a $\epsilon > 0$ there exists a gauge function δ in $\overline{\mathbb{R}}^n$ such that, for each $\delta -$ fine division $\mathcal{D}_\delta$ of $\mathbb{R}^n$, the corresponding Riemann sums satisfy

$$\left| \alpha - \mathcal{D}_\delta \sum h(x, I) \right| < \epsilon.$$

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Definitions in $\mathbb{R}^T$

Let T be an interval in $(0, \infty)$.

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Let us denote $\mathcal{N} = N(T) = \{N = \{t_1, t_2, ..., t_n\} \subset T, n \in \mathbb{N}\}$.
Suppose that $t_1 < t_2 < ... < t_n$. (R.V.)

Definition

$\mathcal{N} = \mathcal{N}(T)$ denotes the class of finite subsets N of T. Let N be a finite set in $\mathcal{N}$, such that $t_1 < t_2 < ... < t_n$.
A cell in $\mathbb{R}^T$ is

$$I[N] = I_1 \times I_2 \times ... \times I_n \times \mathbb{R}^{T \setminus N} = I(N) \times \mathbb{R}^{T \setminus N}.$$

Definition

It is said that $(x, N, I[N])$ *is associated in* $\mathbb{R}^T$ *is the corresponding finite-dimensional pair* $(x(N), I(N))$ *is associated in the finite-dimensional space* $\mathbb{R}^N$.

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definition

A gauge function γ on $\mathbb{R}^T$ is considered as a pair of mappings (δ, L).

Definition

With $N = \{t_1, ..., t_n\} \in \mathcal{N}(T)$, *an associated triple* $(x, N, I[N])$ *is* $\gamma -$ *fine if,*

- $L(x) \subset N$ *and*
- (x_i, I_i) *is* $\delta -$ *fine for* $t_j \in N$, $1 \leq j \leq N$.

Definition

A division of $\mathbb{R}^T$ *is a finite collection* $\mathcal{D}$ *of point-cell pairs* $(x, I[N])$ *such that the corresponding* $(x, N, I[N])$ *are associated, and the cells* $I[N]$ *form a partition* $\mathcal{P}$ *of* $\mathbb{R}^T$*.*

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Definition

A function h of associated triples $(x, N, I[N])$ is integrable on $\mathbb{R}^T$, with integral

$$\alpha = \int_{\mathbb{R}^T} h(x, N, I[N]),$$

if, given $\epsilon > 0$, there exists a gauge γ so that, for each $\gamma -$ fine division $\mathcal{D}_\gamma$ of $\mathbb{R}^T$, the corresponding Riemann sum satisfies

$$|\alpha - (\mathcal{D}_\gamma) \sum h| < \epsilon.$$

Conclusions

- The Feynman path integral cannot be defined in the classical sense because of the integrand is non-absolutely integrable.

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- The Feynman path integral cannot be defined in the classical sense because of the integrand is non-absolutely integrable.
- The Henstock approximation does not require the concept of measure and the absolute value of the integrand does not need to be integrable.

- Equivalence between the Schrödinger equation solutions according to the perturbation theory (Trotter's Theorem) and Henstock Theory.

$$\boxed{\text{const} \int_{\Omega} e^{iS(\omega;t)} \varphi(\omega(t)) \mathcal{D}\omega}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\text{Trotter's Theorem} \qquad\qquad \text{Henstock's Theory}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\boxed{:= \lim_{n\to\infty} \left(K_m^{t/n} M_V^{t/n} \right)^n \varphi(x)} \quad \xleftrightarrow[\text{Equivalent}]{} \quad \boxed{:= \int_{R^{T'}} \mathcal{U}(x_{T'}) G(I[N])}$$

An integrand in $\mathbb{R}^T$ might then take the form

$$h(x, N, I[N]) = f(x)|I[N]|$$

for some point function f, where $|I[N]|$ is given by expression

$$|I[N]| = \begin{cases} \prod_{j=1}^{n}(v_j - u_j) & \text{if } I_j = (u_j, v_j] \\ 0 & \text{otherwise} \end{cases}$$

- Because $\mathbb{R}^T$ is unbounded in each dimension it is easy to see that constant functions $f(x)$ are not integrable on $\mathbb{R}^T$ with respect to $|I[N]|$ unless, for instance, $f(x)$ is equal to zero for all x. That is why this approach will be based on an application of the Henstock integration technique using

$$\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2}\,dx.$$

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$$\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2} dx.$$

does not exist in Lebesgue sense, but it does in Henstock-Kurzweil sense and

$$\int_{-\infty}^{\infty} e^{\frac{i}{2}x^2} dx = \sqrt{\frac{2\pi}{-i}}.$$

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Thanks.