The 2d O(3) Model under Gradient Flow

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Functional integral

Partition function:

$$Z = \langle 0|0 \rangle = \int \mathcal{D}\phi \exp(iS[\phi])$$

Euclidean time: perform Wick rotation $t \to -it$

Connection with statistical mechanics!

Lattice regularization: discretize space-time variables (here: cubic with spacing 1).

Statistical approach

The probability of a configuration $[\phi]$ is given by

$$p[\phi] = \frac{\exp(-S_E[\phi])}{Z}.$$

Measurements on a large set of configurations randomly distributed with that probability (Markov Chain Monte Carlo).

Efficient cluster algorithm available for the Heisenberg model.

The 2d O(3) model (Heisenberg model)

- Square lattice ($V = L \times L$) with periodic boundaries.
- Interactions between nearest neighbours $\langle xy \rangle$.

$$S[\vec{s}] = \beta \sum_{\langle xy \rangle} (1 - \vec{s}_x \cdot \vec{s}_y) \qquad\qquad |\vec{s}_x| = 1.$$

- Each site has a classical spin $\vec{s}_x \in \mathbb{S}^2$.
- Global O(3) symmetry.

Why this model?

Shares basic properties with QCD:

- Asymptotic freedom
- Dynamically generated mass gap
- Topological sectors

Correlation length ξ

The correlation length ξ measures the decay of the correlation between well separated spins in the lattice ($|x - y| \gg 1$).

Notice that ξ is the physical scale of the system.

In a large volume ($V \to \infty$):

$$\langle \vec{s}_x \cdot \vec{s}_y \rangle \propto \exp\left(-\frac{|x-y|}{\xi} \right).$$

In finite volumes with periodic boundaries:

$$\langle \vec{s}_x \cdot \vec{s}_y \rangle \propto \cosh\left(\frac{|x-y| - L/2}{\xi} \right).$$

Setting the scale (lattice units)

To suppress finite-size effects: $\xi \ll L$.

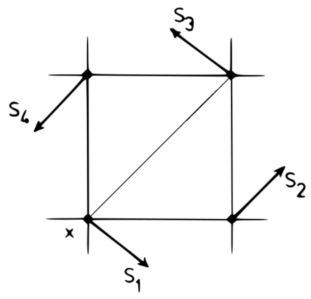
We work with configurations close to $L/\xi \simeq 6$ (pretty safe)

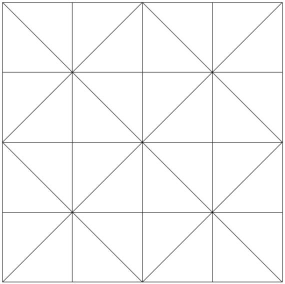
To suppress lattice artifacts: $1 \ll \xi$.

Since $\xi = \xi(\beta, L)$ fine tuning is required.

Topological charge on the lattice

We use the geometric definition (Berg and Lüscher 1981):





- Each plaquette in the lattice is split in two triangles t_{xyz}.
- The oriented solid angle spanned by the spins on their vertices is called A_{xyz}.

$$Q[\vec{s}] = \sum_x q_x = \frac{1}{4\pi} \sum_{t_{xyz}} A_{xyz} \in \mathbb{Z}.$$

Topological susceptibility χ_t

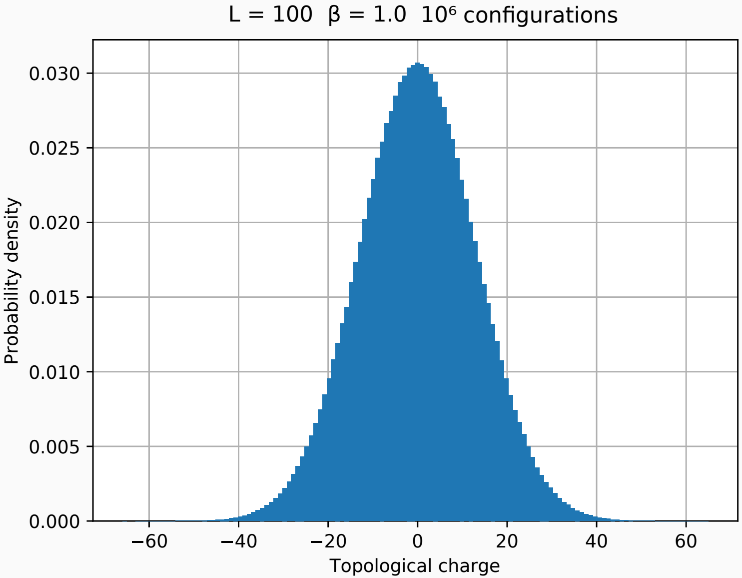
Charge distribution approximately Gaussian.

$$p(Q) \propto \exp\left(-\frac{Q^2}{2\chi_t V}\right)$$

$$\text{mean} \rightarrow \langle Q \rangle = 0 \qquad\qquad \text{variance} \rightarrow \chi_t V$$

$$\chi_t = \frac{\langle Q^2 \rangle - \cancelto{0}{\langle Q \rangle^2}}{V} = \frac{\langle Q^2 \rangle}{V}.$$

Topological charge distribution



Problem with the topology in the continuum limit

Exceptional small plaquettes with charge different from zero are called dislocations.

Dislocations have strong influence on the susceptibility.

The scaling quantity $\chi_t \xi^2$ (dimensionless) diverges in the continuum limit $\xi \to \infty$[1].

Can we handle dislocations?

[1] Lüscher 1982; Blatter et al. 1996.

Gradient Flow[2]

- Continuous smoothing operation.
- Corresponds to a renormalization group transformation.
- Supposed to eliminate dislocations while preserving large instantons.

[2] Lüscher 2010.

Gradient Flow on the lattice

Flow equations for 2d O(3) model in the continuum[3]:

$$\partial_t s^i(t,\vec{x}) = P^{ij}(t,\vec{x})\partial_\mu\partial_\mu s^j(t,\vec{x}),$$
$$P^{ij}(t,\vec{x}) = \delta^{ij} - s^i(t,\vec{x})s^j(t,\vec{x}).$$

On the lattice we use centralized discretization:

$$\vec{s}(t,\vec{x}) \longrightarrow \vec{s}_{x_1,x_2}(t),$$

$$\begin{aligned}\partial_\mu\partial_\mu s^j(t,\vec{x}) \longrightarrow\ & s^j_{x_1+1,x_2}(t) + s^j_{x_1-1,x_2}(t) \\ & + s^j_{x_1,x_2+1}(t) + s^j_{x_1,x_2-1}(t) \\ & - 4s^j_{x_1,x_2}(t).\end{aligned}$$

[3] Makino and Suzuki 2015.

Effects of Flow on topology

Is topology well defined in this model?

We want to see whether the effect of the gradient flow on the topology allows the quantity $\chi_t \xi^2$ to converge when $\xi \to \infty$.

Scale for the Gradient Flow time

It is useful to consider

$$\langle e \rangle := \frac{\langle S \rangle}{\beta L^2}.$$

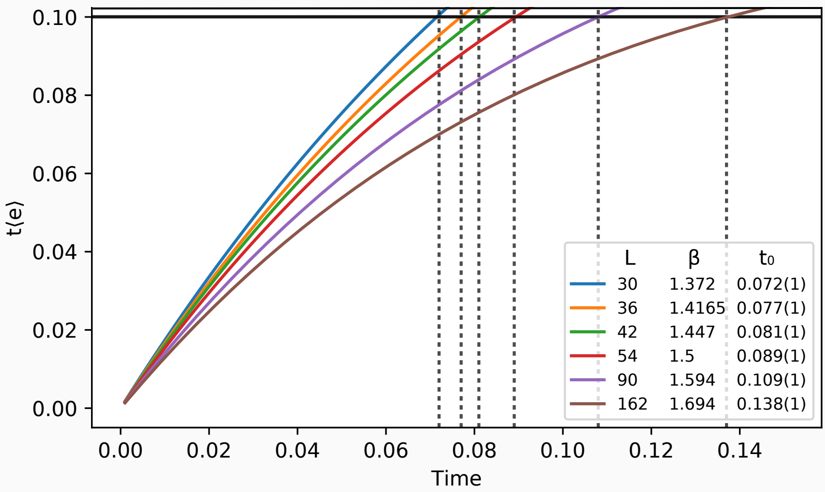
Notice that $t\langle e \rangle$ is dimensionless.

The *flow-time unit* t_0 satisfies:

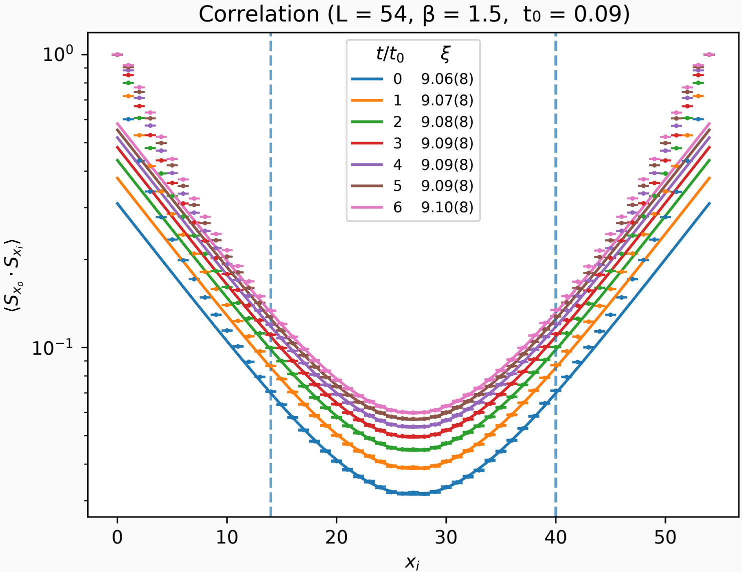
$$t_0 \langle e \rangle = 0.1.$$

We analyze the effects of gradient flow at multiples of t_0.

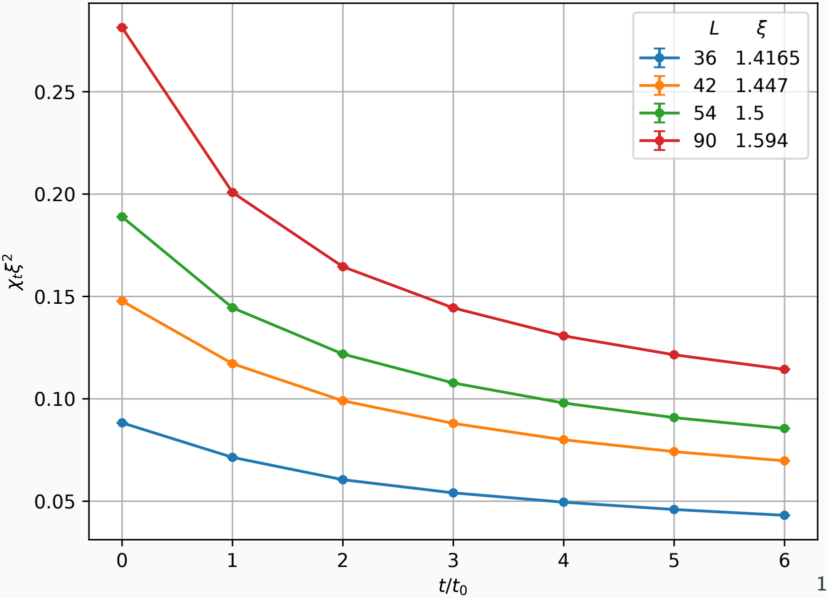
Flow time scale



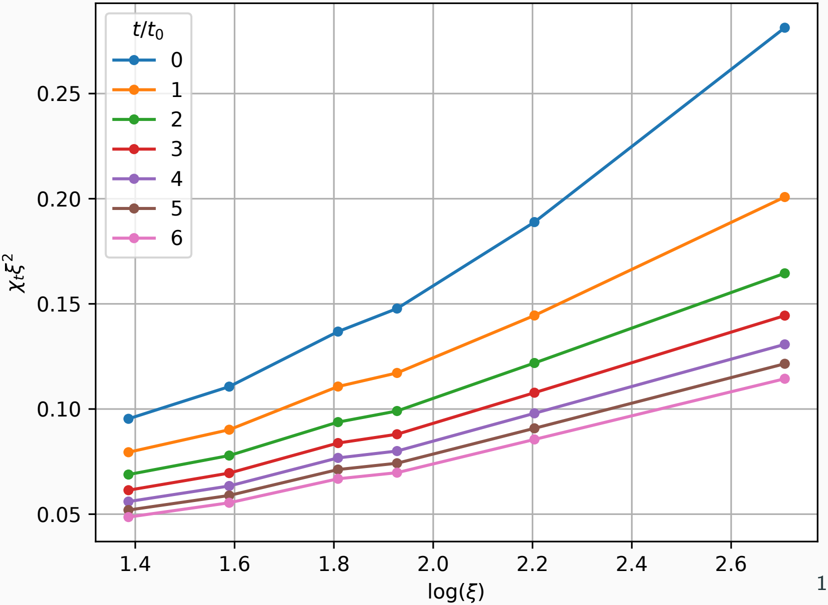
Correlation



Evolution of $\chi_t \xi^2$ under Gradient Flow



Scaling of $\chi_t\xi^2$ towards the continuum limit



Conclusions (work in progress...)

- We study the Heisenberg model 2d O(3).
- Does this model have a well defined topology?
- The established answer is no.
- Gradient Flow could change the fate of the model.

References

Berg, B and Martin Lüscher (1981). "Definition and statistical distributions of a topological number in the lattice O (3) σ-model". In: *Nuclear Physics B* 190.2, pp. 412–424.

Bietenholz, Wolfgang (2016). "Hadron physics from lattice QCD". In: *International Journal of Modern Physics E* 25.07, p. 1642008.

Bietenholz, Wolfgang et al. (2010). "Topological lattice actions". In: *Journal of High Energy Physics* 2010.12, pp. 1–36.

Bietenholz, Wolfgang et al. (2011). "Quantum Field Theories on the Lattice: Concepts behind their Numerical Simulations". In: *AIP Conference Proceedings*. Vol. 1361. 1. AIP, pp. 11–29.

Blatter, Marc et al. (1996). “Instantons and the fixed point topological charge in the two-dimensional O (3) σ model”. In: *Physical Review D* 53.2, p. 923.

Lüscher, Martin (1982). “Does the topological susceptibility in lattice σ models scale according to the perturbative renormalization group?” In: *Nuclear Physics B* 200.1, pp. 61–70.

– (2010). “Properties and uses of the Wilson flow in lattice QCD”. In: *Journal of High Energy Physics* 2010.8, pp. 1–18.

Makino, Hiroki and Hiroshi Suzuki (2015). “Renormalizability of the gradient flow in the 2D O (N) non-linear sigma model”. In: *Progress of Theoretical and Experimental Physics* 2015.3, 033B08.