Small x Asymptotics of the Quark and Gluon Helicity Distributions

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work with Dan Pitonyak and Matt Sievert,

arXiv:1706.04236 [nucl-th] and 5 other papers

Outline

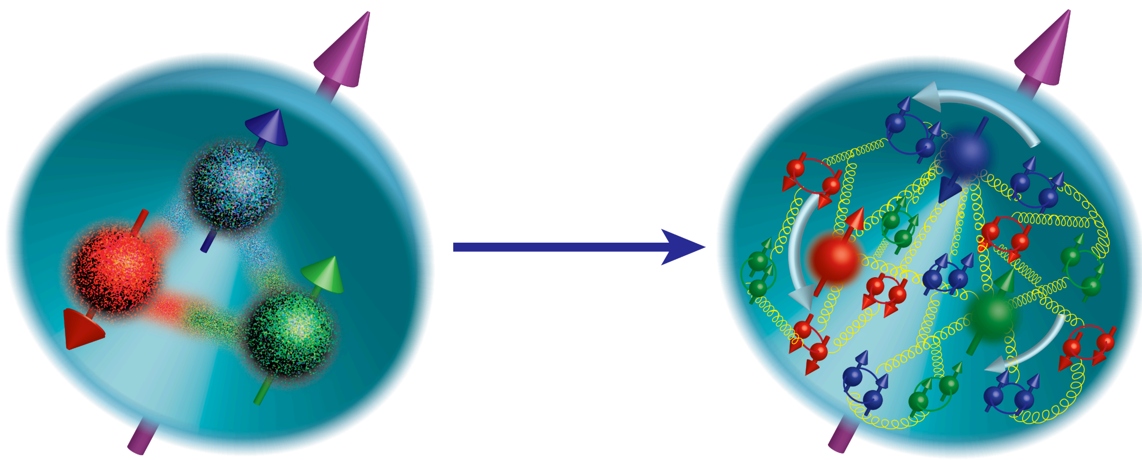
- Goal: understanding the proton spin coming from small x partons
- Quark Helicity:
  - Quark helicity distribution at small x
  - Small-x evolution equations for quark helicity
  - Small-x asymptotics of quark helicity
- Gluon Helicity:
  - Gluon helicity distribution at small x
  - Small-x evolution equations for gluon helicity
  - Small-x asymptotics of quark helicity TMDs
- Main results (at large N_c):

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x} \right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \tfrac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s \, N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s \, N_c}{2\pi}}$$

Our Goal: Proton Spin at Small x

Proton Spin

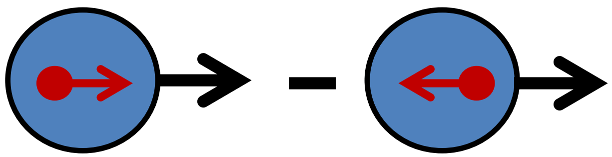


Our understanding of nucleon spin structure has evolved:

- In the 1980's the proton spin was thought of as a sum of constituent quark spins (left panel)
- Currently we believe that the proton spin is a sum of the spins of valence and sea quarks and of gluons, along with the orbital angular momenta of quarks and gluons (right panel)

Helicity Distributions

- To quantify the contributions of quarks and gluons to the proton spin one defines helicity distribution functions: number of quarks/gluons with spin parallel to the proton momentum minus the number of quarks/gluons with the spin opposite to the proton momentum:



- The helicity parton distributions are

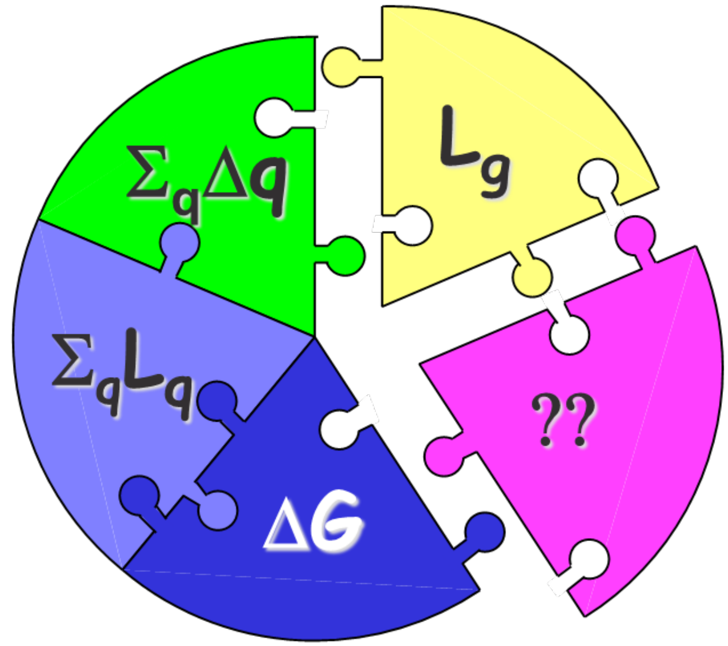
$$\Delta f(x,Q^2) \equiv f^+(x,Q^2) - f^-(x,Q^2)$$

with the net quark helicity distribution

$$\Delta\Sigma \equiv \Delta u + \Delta\bar{u} + \Delta d + \Delta\bar{d} + \Delta s + \Delta\bar{s}$$

and $\Delta G(x, Q^2)$ the gluon helicity distribution.

Proton Helicity Sum Rule



- Helicity sum rule:

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

with the net quark and gluon spin

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int_0^1 dx\, \Delta G(x, Q^2)$$

- L_q and L_g are the quark and gluon orbital angular momenta

Proton Spin Puzzle

$$S_q(Q^2) = \frac{1}{2} \int\limits_0^1 dx\, \Delta\Sigma(x, Q^2)$$

- The spin puzzle began when the EMC collaboration measured the proton g_1 structure function ca 1988. Their data resulted in

$$\Delta\Sigma \approx 0.1 \div 0.2$$

- It appeared quarks do not carry all of the proton spin (which would have corresponded to $\Delta\Sigma = 1$).

- Missing spin can be

$$\frac{1}{2} = S_q + L_q + S_g + L_g$$

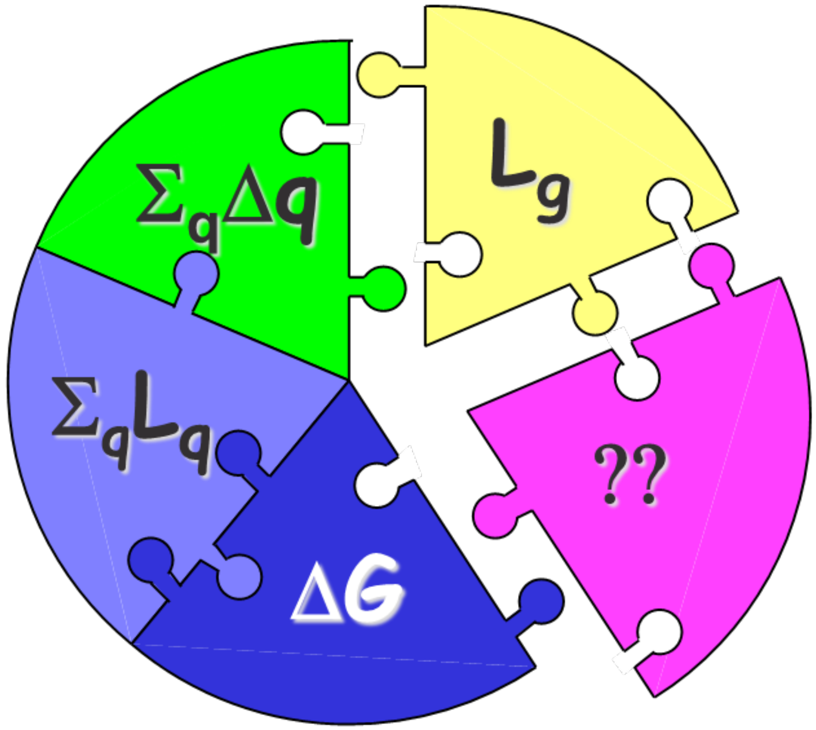
  - Carried by gluons
  - In the orbital angular momenta of quarks and gluons
  - At small x:

$$S_q(Q^2) = \frac{1}{2} \int\limits_0^1 dx\, \Delta\Sigma(x, Q^2) \qquad S_g(Q^2) = \int\limits_0^1 dx\, \Delta G(x, Q^2)$$

Can't integrate down to zero, use x_{min} instead!

  - Or all of the above!

Proton Spin Pie Chart



- The proton spin carried by the quarks is estimated to be (for $0.001 < x < 1$)

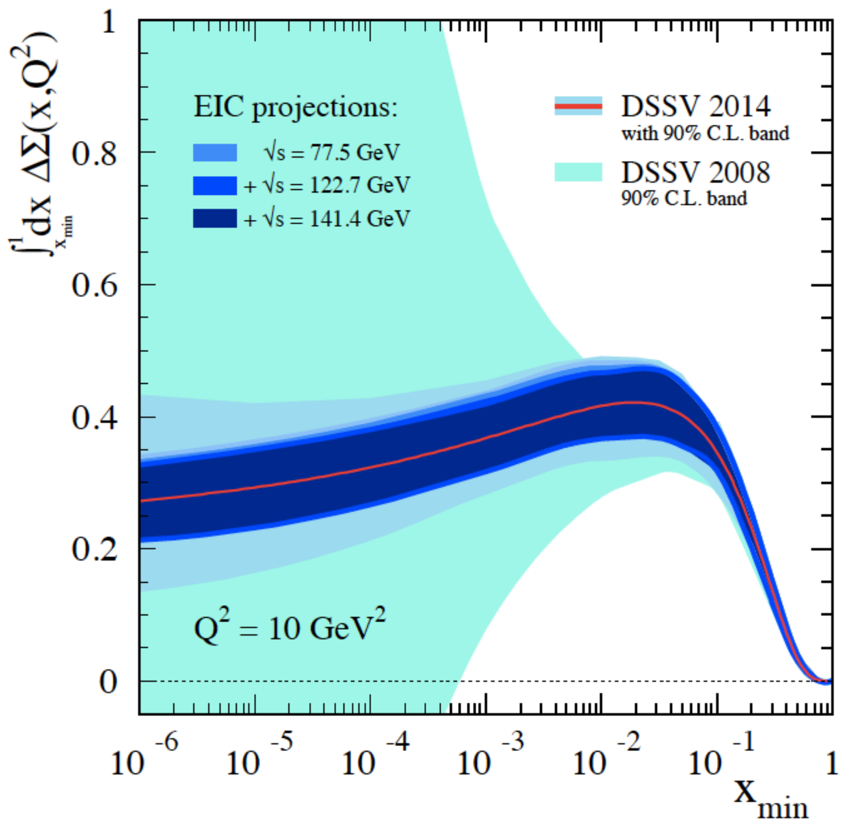
$$S_q(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.15 \div 0.20$$

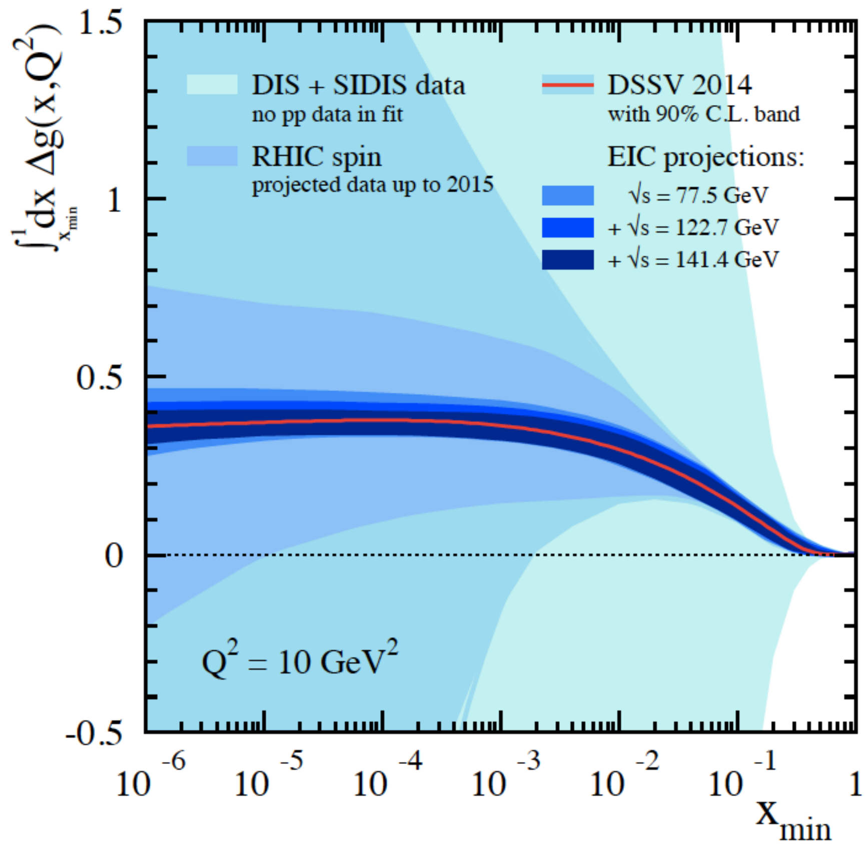
- The proton spin carried by the gluons is (for $0.05 < x < 1$)

$$S_G(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.13 \div 0.26$$

- Unfortunately the uncertainties are large. Note also that the x-ranges are limited, with more spin (positive or negative) possible at small x.

How much spin is at small x?

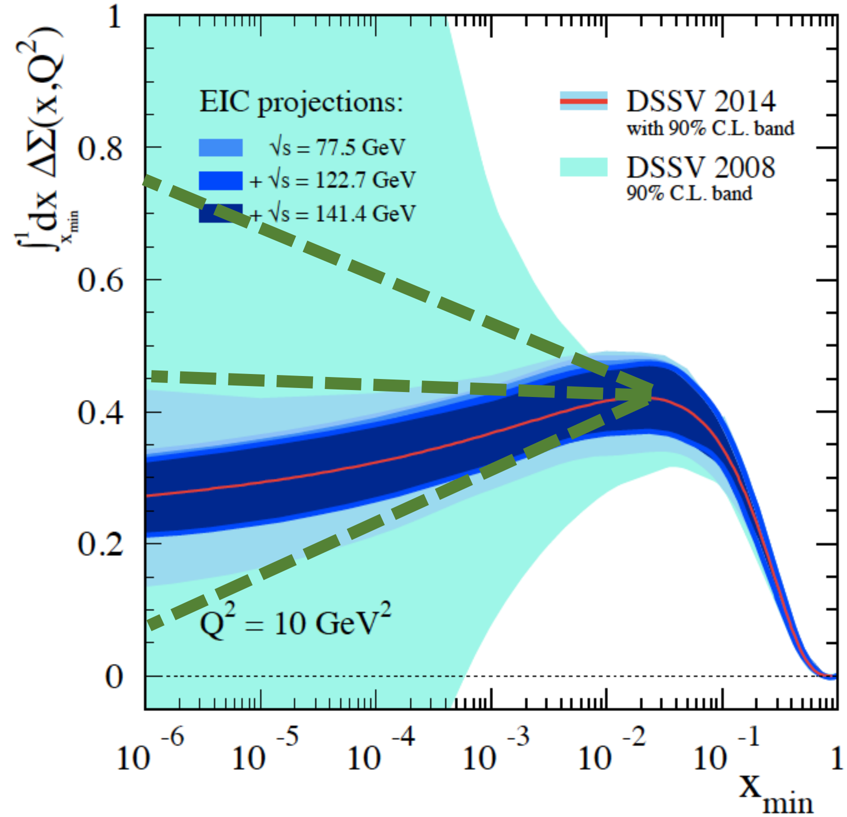




- E. Aschenaur et al, arXiv:1509.06489 **[hep-ph]**
- Uncertainties are very large at small x!

Spin at small x

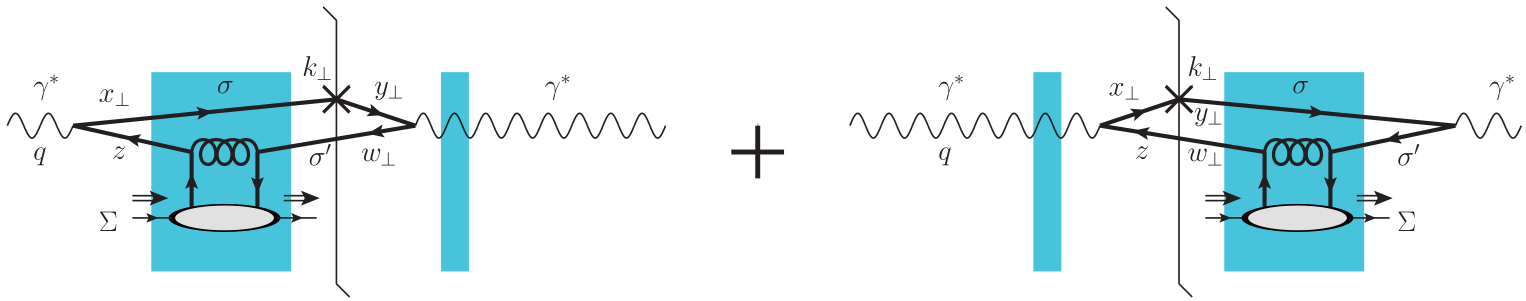
- The goal of this project is to provide theoretical understanding of helicity PDF's at very small x.
- Our work would provide guidance for future hPDF's parametrizations of the existing and new data (e.g., the data to be collected at EIC).
- Alternatively the data can be analyzed using our small-x evolution formalism.



Quark Helicity Evolution at Small x
flavor-singlet case

Yu.K., M. Sievert, arXiv:1505.01176 [hep-ph]
Yu.K., D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph],
arXiv:1610.06197 [hep-ph], arXiv:1610.06188 [hep-ph],
arXiv:1703.05809 [hep-ph]

Quark Helicity Observables at Small x



- One can show that the g_1 structure function and quark helicity PDF (Δq) and TMD at small-x can be expressed in terms of the polarized dipole amplitude (flavor singlet case):

$$g_1^S(x, Q^2) = \frac{N_c\, N_f}{2\,\pi^2 \alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2, z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2, z)} \right] G(x_{01}^2, z),$$

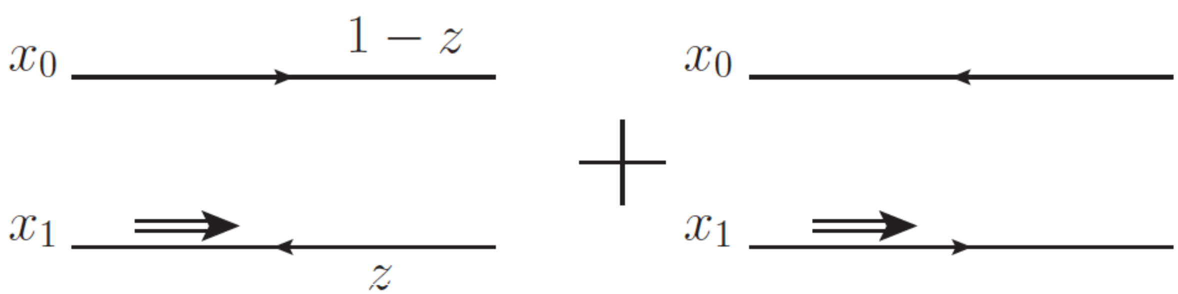
$$\Delta q^S(x, Q^2) = \frac{N_c\, N_f}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2} G(x_{01}^2, z),$$

$$g_{1L}^S(x, k_T^2) = \frac{8\, N_c\, N_f}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2 x_{01}\, d^2 x_{0'1}\, e^{-i\underline{k}\cdot(\underline{x}_{01} - \underline{x}_{0'1})} \frac{\underline{x}_{01} \cdot \underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2} G(x_{01}^2, z)$$

- Here s is cms energy squared, $z_i = \Lambda^2/s$, $G(x_{01}^2, z) \equiv \int d^2 b\, G_{10}(z)$

Polarized Dipole

- All flavor singlet small-x helicity observables depend on one object, “polarized dipole amplitude”:



$$G_{10}(z) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \text{tr}\left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger}\right] + \text{tr}\left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger}\right] \right\rangle\!\!\right\rangle(z)$$

unpolarized quark

polarized quark (“polarized Wilson line”): eikonal propagation, non-eikonal spin-dependent interaction

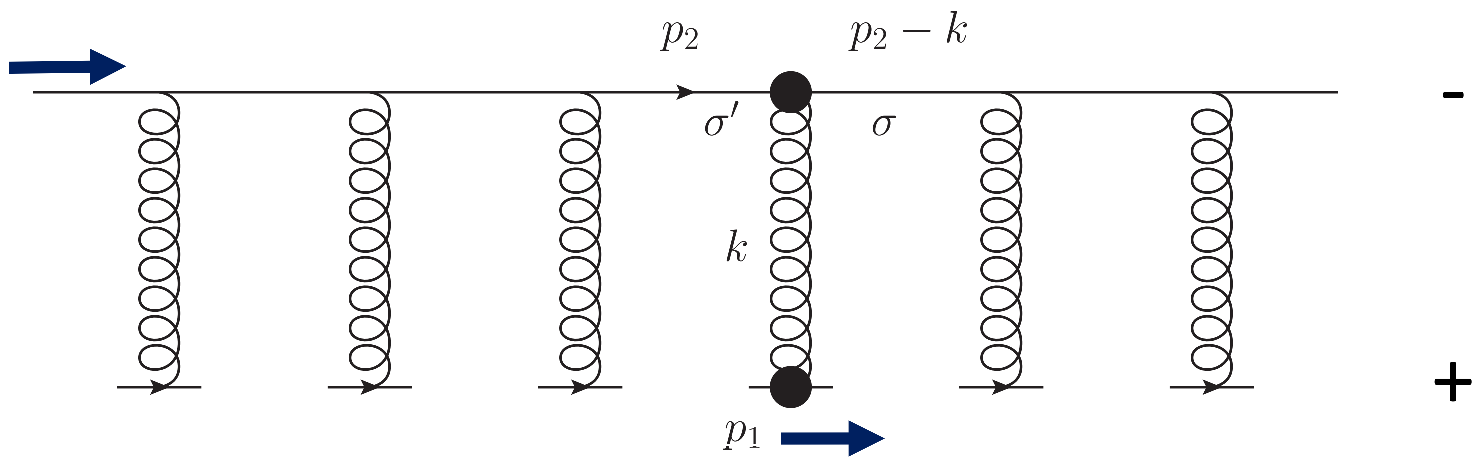
$$V_{\underline{x}} \equiv \mathcal{P} \exp\left[ig \int\limits_{-\infty}^{\infty} dx^+ \, A^-(x^+, 0^-, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle(z) \equiv zs \left\langle \mathcal{O} \right\rangle(z)$$

“Polarized Wilson line”

To obtain an explicit expression for the “polarized Wilson line” operator due to a sub-eikonal gluon exchange (as opposed to the sub-eikonal quarks exchange), consider multiple Coulomb gluon exchanges with the target:



Most gluon exchanges are eikonal spin-independent interactions, while one of them is a spin-dependent sub-eikonal exchange. (cf. Mueller ‘90, McLerran, Venugopalan ‘93)

“Polarized Wilson line”

- A simple calculation in A^-=0 gauge yields the “polarized Wilson line”:

$$V_{\underline{x}}^{pol} = \frac{1}{2\,s} \int\limits_{-\infty}^{\infty} dx^- \, \text{P} \exp \left\{ ig \int\limits_{x^-}^{\infty} dx'^- \, A^+(x'^-, \underline{x}) \right\} \, ig \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, \text{P} \exp \left\{ ig \int\limits_{-\infty}^{x^-} dx'^- \, A^+(x'^-, \underline{x}) \right\}$$

where $\underline{A}_\Sigma(x^-, \underline{x}) = \frac{\Sigma}{2p_1^+} \, \underline{\tilde{A}}(x^-, \underline{x})$

is the spin-dependent sub-eikonal gluon field of the plus-direction moving target with helicity Σ.

(A^+ is the unpolarized eikonal field.)

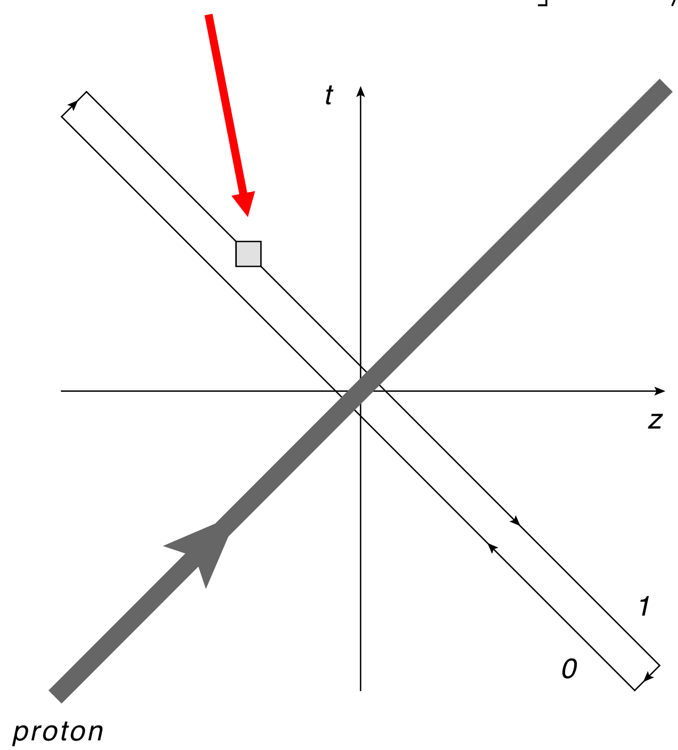
Polarized Dipole Amplitude

- The polarized dipole amplitude is then defined by

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \, \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

with the standard light-cone Wilson line

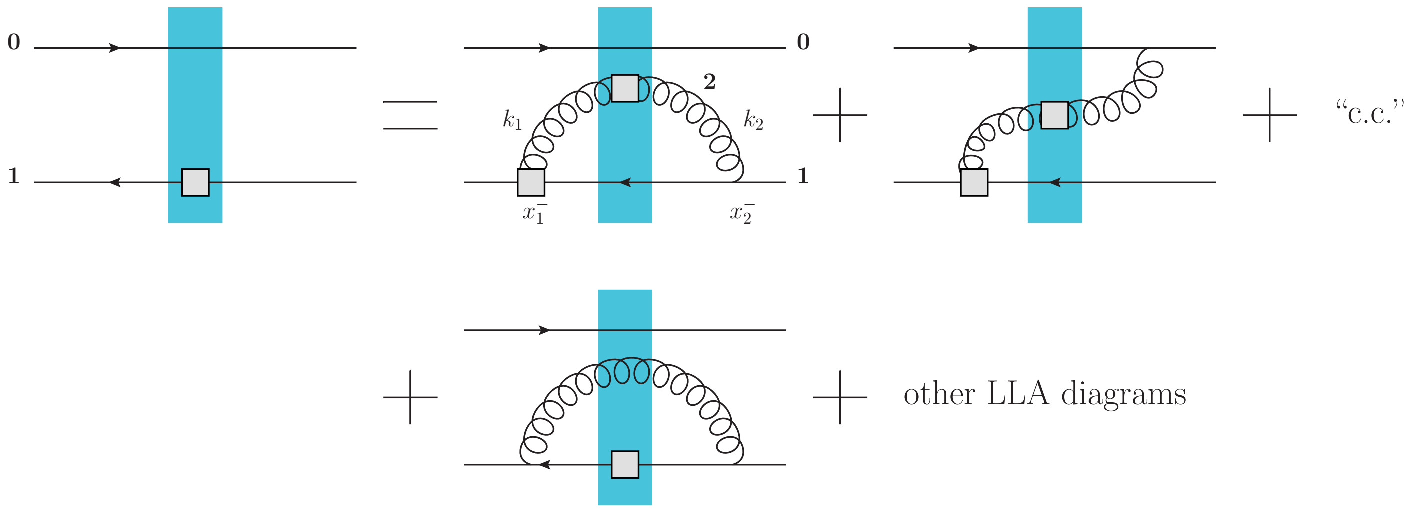
$$V_{\underline{x}}[b^-, a^-] = \mathrm{P} \exp \left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$



Quark Helicity TMDs: Small-x Evolution

Evolution for Polarized Quark Dipole

- We can evolve the polarized dipole operator and obtain its small-x evolution equation:

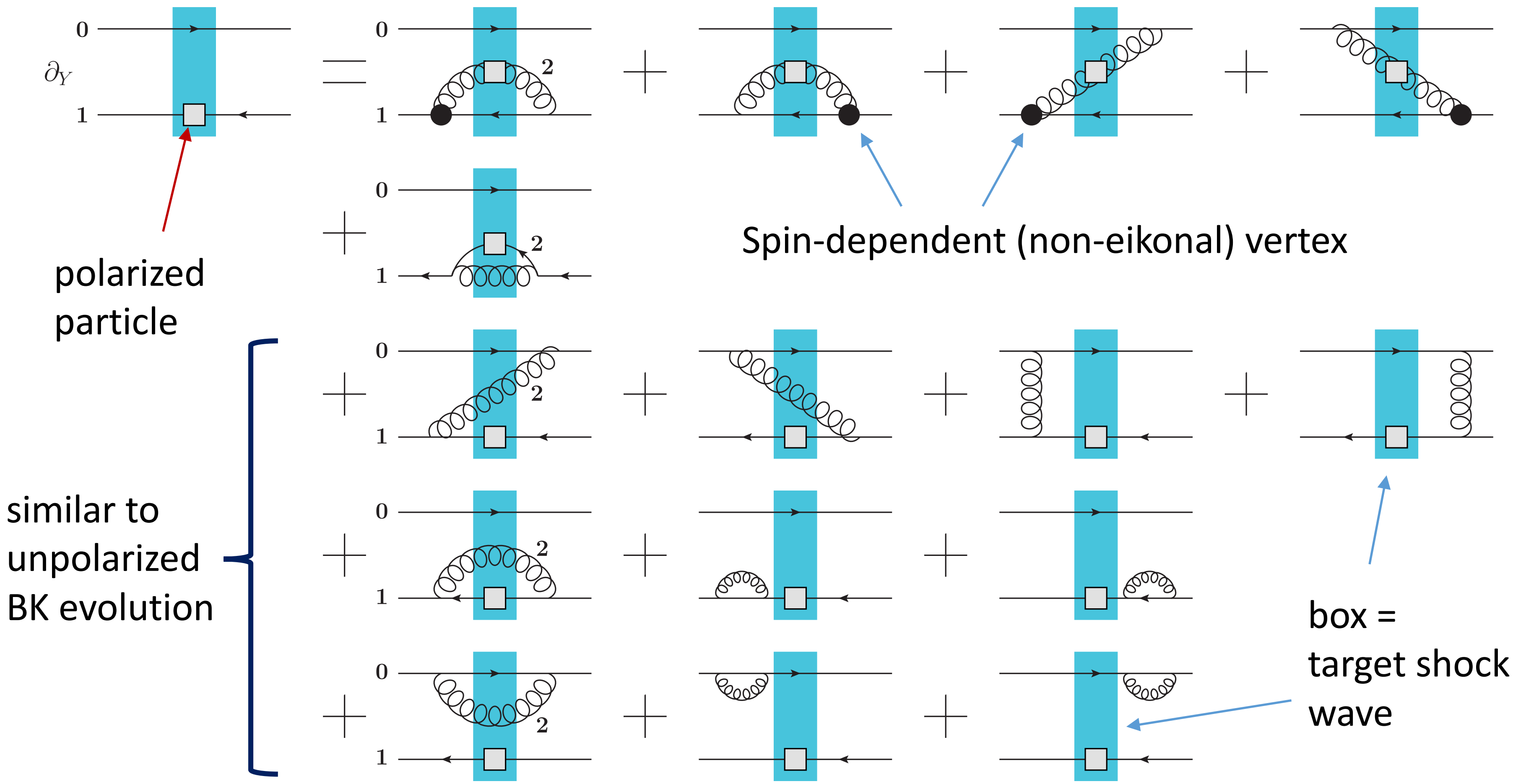


- From the first two graphs on the right we get

$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s}{\pi^2} \int^{z} \frac{dz'}{z'} \int \frac{d^2 x_2}{x_{21}^2} \frac{1}{N_c} \left\langle\!\!\left\langle \text{tr} \left[t^b V_0 t^a V_1^\dagger \right] U_2^{pol\, ba} \right\rangle\!\!\right\rangle + \ldots$$

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Resummation Parameter

- For helicity evolution the resummation parameter is different from BFKL, BK or JIMWLK, which resum powers of leading logarithms (LLA)

$$\alpha_s \ \ln(1/x)$$

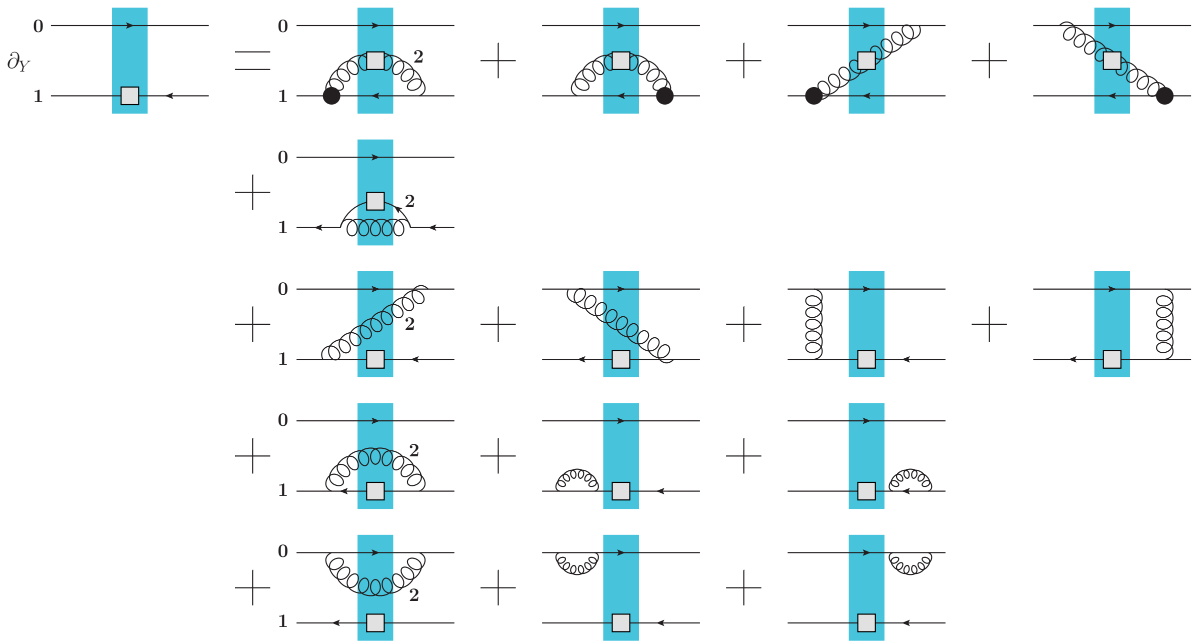
- Helicity evolution resummation parameter is double-logarithmic (DLA):

$$\alpha_s \ \ln^2 \frac{1}{x}$$

- The second logarithm of x arises due to transverse momentum (or transverse coordinate) integration being logarithmic both in UV and IR.

- This was known before: Kirschner and Lipatov ’83; Kirschner ’84; Bartels, Ermolaev, Ryskin ‘95, ‘96; Griffiths and Ross ’99; Itakura et al ’03; Bartels and Lublinsky ‘03.

Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = \frac{1}{z\, s} \langle \ldots \rangle$$

$$\rho'^{\,2} = \frac{1}{z'\, s}$$

$$\frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle (z) = \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0 (z) + \frac{\alpha_s}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2}$$

$$\times \left\{ \theta(x_{10} - x_{21})\, \frac{2}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_1^{unp\,\dagger}\right]\, U_2^{pol\, ba} \right\rangle\!\!\right\rangle (z') \right.$$

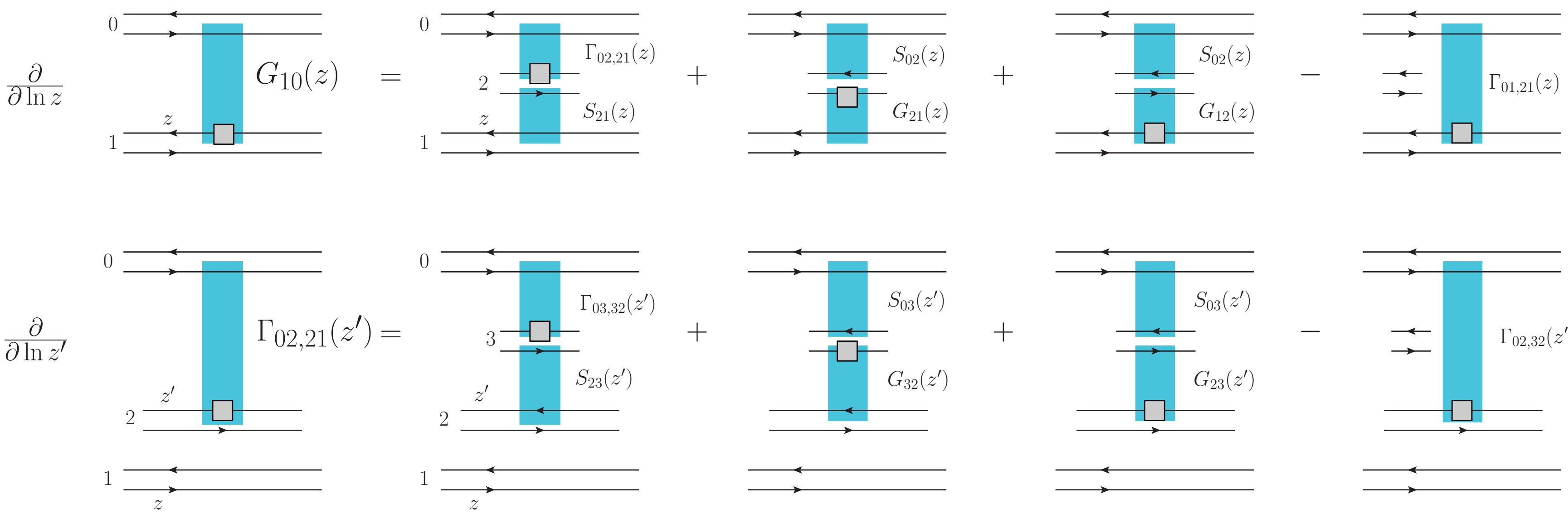
$$+ \theta(x_{10}^2 z - x_{21}^2 z')\, \frac{1}{N_c} \left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_0^{unp}\, t^a\, V_2^{pol\,\dagger}\right]\, U_1^{unp\, ba} \right\rangle\!\!\right\rangle (z')$$

$$\left. + \theta(x_{10} - x_{21})\, \frac{1}{N_c} \left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_2^{unp\,\dagger}\right]\, \mathrm{tr}\left[V_2^{unp}\, V_1^{pol\,\dagger}\right] \right\rangle\!\!\right\rangle (z') - N_c \left\langle\!\!\left\langle \mathrm{tr}\left[V_0^{unp}\, V_1^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle (z') \right] \right\}$$

Equation does not close!

Polarized Dipole Evolution in the Large-N_c Limit

In the large-N_c limit the equations close, leading to a system of 2 equations:



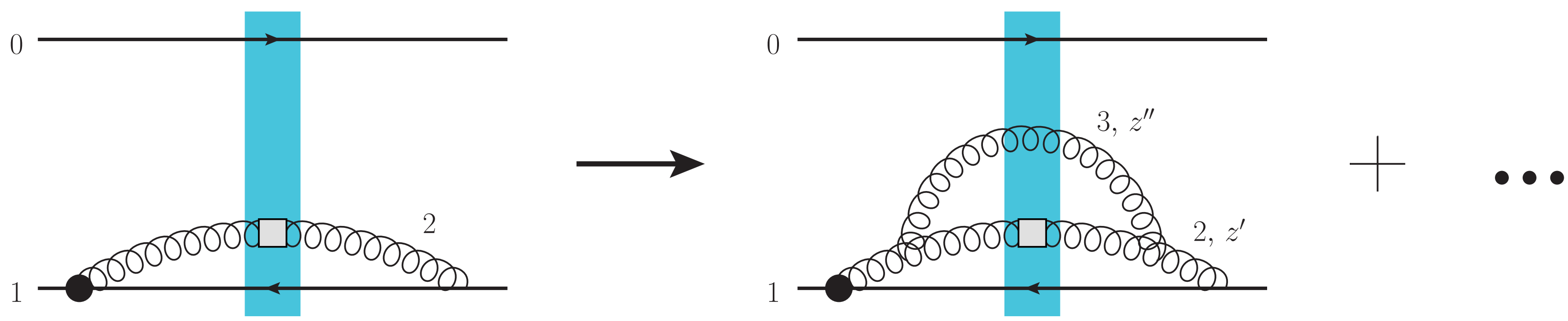
$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[2\,\Gamma_{02,\,21}(z')\,S_{21}(z') + 2\,G_{21}(z')\,S_{02}(z') + G_{12}(z')\,S_{02}(z') - \Gamma_{01,\,21}(z')\right]$$

$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2\, z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[2\,\Gamma_{03,\,32}(z'')\,S_{23}(z'') + 2\,G_{32}(z'')\,S_{03}(z'') + G_{23}(z'')\,S_{03}(z'') - \Gamma_{02,\,32}(z'')\right]$$

S = found from BK/JIMWLK, it is LLA

Your friendly "neighborhood" dipole

- There is a new object in the evolution equation – **the neighbor dipole**.
- This is specific for the DLA evolution. Gluon emission may happen in one dipole, but, due to transverse distance ordering, may 'know' about another dipole:



$$x_{21}^2\, z' \gg x_{32}^2\, z''$$

- We denote the evolution in the neighbor dipole 02 by $\Gamma_{02,\,21}(z')$

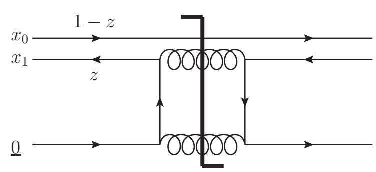
Large-N_c Evolution

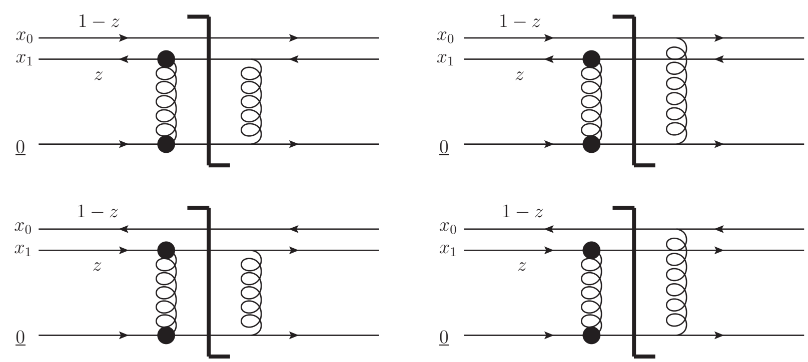
- In the strict DLA limit (S=1) and at large N_c we get (here Γ is an auxiliary function we call the 'neighbour dipole amplitude')

$$G(x_{10}^2, z) = G^{(0)}(x_{10}^2, z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z') + 3\, G(x_{21}^2, z')\right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z') = \Gamma^{(0)}(x_{10}^2, x_{21}^2, z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\right\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z'') + 3\, G(x_{32}^2, z'')\right]$$

- The initial conditions are given by the Born-level graphs





$$\Gamma^{(0)}(x_{10}^2, x_{21}^2, z) = G^{(0)}(x_{10}^2, z)$$

$$G^{(0)}(x_{10}^2, z) = \frac{\alpha_s^2 C_F}{N_c} \pi \left[C_F \ln \frac{zs}{\Lambda^2} - 2\, \ln(zs\, x_{10}^2)\right]$$

Quark Helicity TMDs: Small-x Asymptotics

Prior Results

- Small-x DLA evolution for the g_1 structure function was first considered by Bartels, Ermolaev and Ryskin (BER) in '96.

- Including the mixing of quark and gluon ladders, they obtained

$$\Delta\Sigma \sim g_1 \sim \left(\frac{1}{x}\right)^{z_s \sqrt{\frac{\alpha_S \, N_C}{2\,\pi}}}$$

  with z_s = 3.45 for 4 quark flavors and z_S=3.66 for pure glue.

$$S_q(Q^2) = \frac{1}{2} \int\limits_0^1 dx \, \Delta\Sigma(x, Q^2)$$

- The power is large: it becomes larger than 1 for realistic strong coupling of the order of α_s = 0.2 − 0.3, resulting in polarized PDFs which actually grow with decreasing x fast enough for the integral of the PDFs over the low-x region to be (potentially) large (infinite).

Numerical Solution

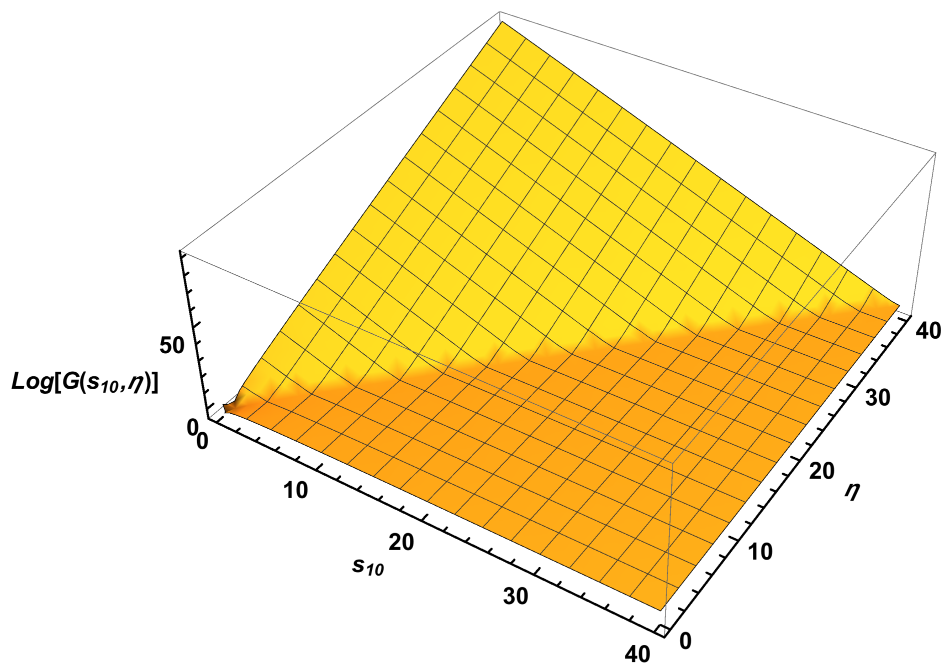
- We discretize the equations

$$G_{ij} = G_{ij}^{(0)} + \Delta\eta\,\Delta s \sum_{j'=i}^{j-1} \sum_{i'=i}^{j'} \left[\Gamma_{ii'j'} + 3\,G_{i'j'}\right],$$

$$\Gamma_{ikj} = \Gamma_{ikj}^{(0)} + \Delta\eta\,\Delta s \sum_{j'=i}^{j-1} \sum_{i'=\max\{i,k+j'-j\}}^{j'} \left[\Gamma_{ii'j'} + 3\,G_{i'j'}\right]$$

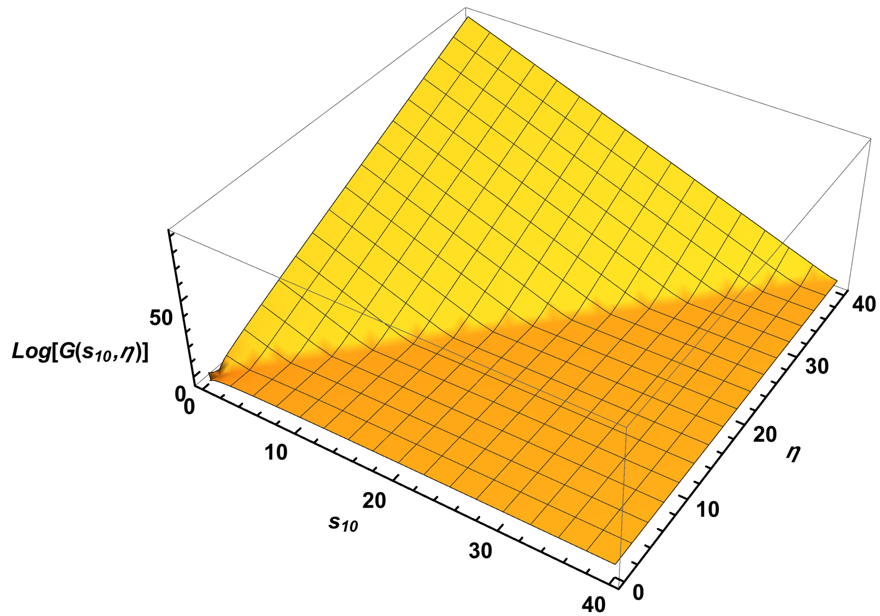
and solve them by progressively populating each fixed-η row in s.

- The solution for G looks like this:



$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}}\,\ln\frac{zs}{\Lambda^2} \qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}}\,\ln\frac{1}{x_{10}^2\,\Lambda^2}$$

Solution of the large-N_C Equations



$$\alpha_h \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- The resulting small-x asymptotics is

$$g_1^S(x, Q^2) \sim \Delta q^S(x, Q^2) \sim g_{1L}^S(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h} \approx \left(\frac{1}{x}\right)^{2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

- Our result, 2.31, is about 35% smaller than BER's 3.66 any-N_C pure glue.

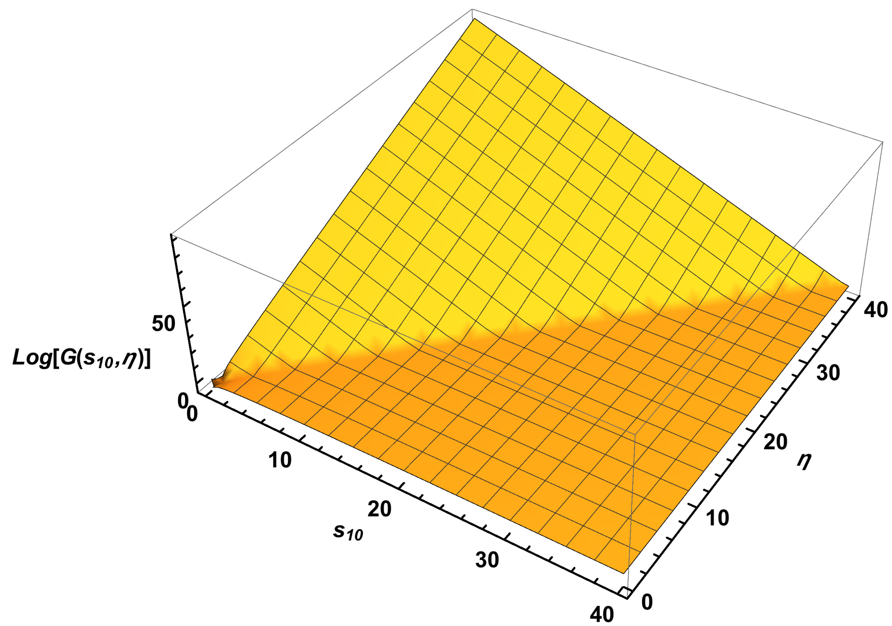
Scaling

- Our numerical solution has a scaling property!

$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2} \qquad s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

- The solution is well approximated by

$$G(s_{10}, \eta) \propto e^{2.31\,(\eta - s_{10})}$$



- This motivated us to look for the solution in the following scaling form:

$$G(s_{10}, \eta) = G(\eta - s_{10})$$
$$\Gamma(s_{10}, s_{21}, \eta') = \Gamma(\eta' - s_{10}, \eta' - s_{21})$$

Scaling Equations

- The large-N_c evolution equations can be rewritten in terms of the scaling variables (not a trivial property, does not work for the large-N_c&N_f equations):

$$G(\zeta) = 1 + \int\limits_0^\zeta d\xi \int\limits_0^\xi d\xi' \ \left[\Gamma(\xi, \xi') + 3\, G(\xi')\right],$$

$$\Gamma(\zeta, \zeta') = 1 + \int\limits_0^{\zeta'} d\xi \int\limits_0^\xi d\xi' \ \left[\Gamma(\xi, \xi') + 3\, G(\xi')\right]$$

$$+ \int\limits_{\zeta'}^{\zeta} d\xi \int\limits_0^{\zeta'} d\xi' \left[\Gamma(\xi, \xi') + 3\, G(\xi')\right]$$

- For simplicity, pick the following initial conditions:

$$G(0) = 1, \quad \Gamma(\zeta', \zeta') = G(\zeta')$$

Analytic Solution

- These scaling equations can be solved exactly via Laplace transform + a few clever tricks, yielding

$$G(\zeta) = \int \frac{d\omega}{2\pi i}\, e^{\omega\,\zeta + \frac{\zeta}{\omega}}\, \frac{\omega^2 - 1}{\omega\,(\omega^2 - 3)},$$

$$\Gamma(\zeta,\zeta') = 4\int \frac{d\omega}{2\pi i}\, e^{\omega\,\zeta' + \frac{\zeta}{\omega}}\, \frac{\omega^2 - 1}{\omega\,(\omega^2 - 3)} - 3\int \frac{d\omega}{2\pi i}\, e^{\omega\,\zeta' + \frac{\zeta'}{\omega}}\, \frac{\omega^2 - 1}{\omega\,(\omega^2 - 3)}.$$

- As usual, the high-energy asymptotics is given by the right-most pole in the complex ω-plane: the pole is at $\omega = +\sqrt{3}$.

Analytic Solution and Intercept

- The (dominant part of the) scaling solution is

$$G(\zeta) \approx \frac{1}{3} e^{\frac{4}{\sqrt{3}}\zeta}$$

$$\Gamma(\zeta, \zeta') \approx \frac{1}{3} e^{\frac{4}{\sqrt{3}}\zeta'} \left(4 e^{\frac{\zeta - \zeta'}{\sqrt{3}}} - 3 \right)$$

$$= G(\zeta') \left(4 e^{\frac{\zeta - \zeta'}{\sqrt{3}}} - 3 \right)$$

- The corresponding helicity intercept is

$$\alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.3094 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- This is in complete agreement with the numerical solution!

$$\alpha_h^q \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

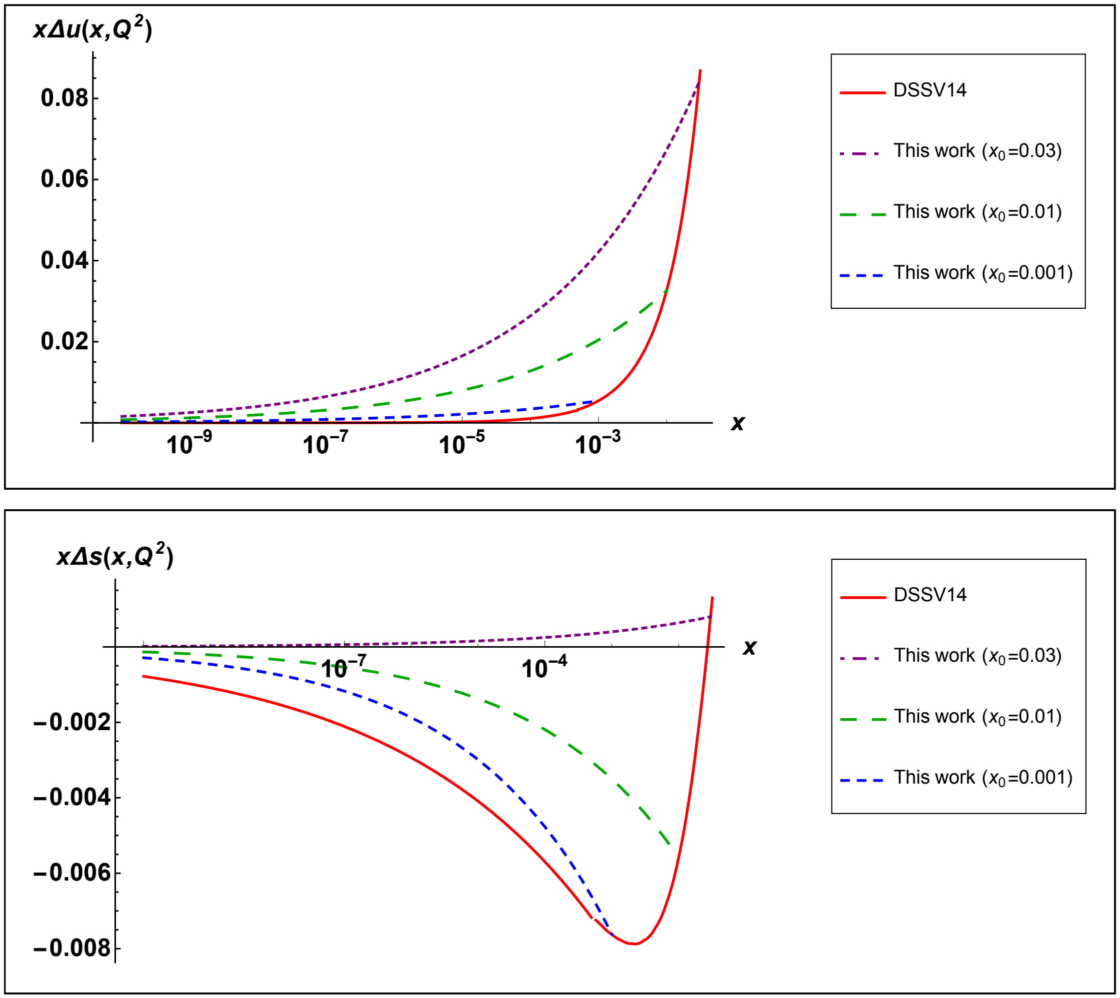
Quark Helicity at Small x

- The small-x asymptotics of quark helicity is (at large N_c)

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Impact of our $\Delta\Sigma$ on the proton spin

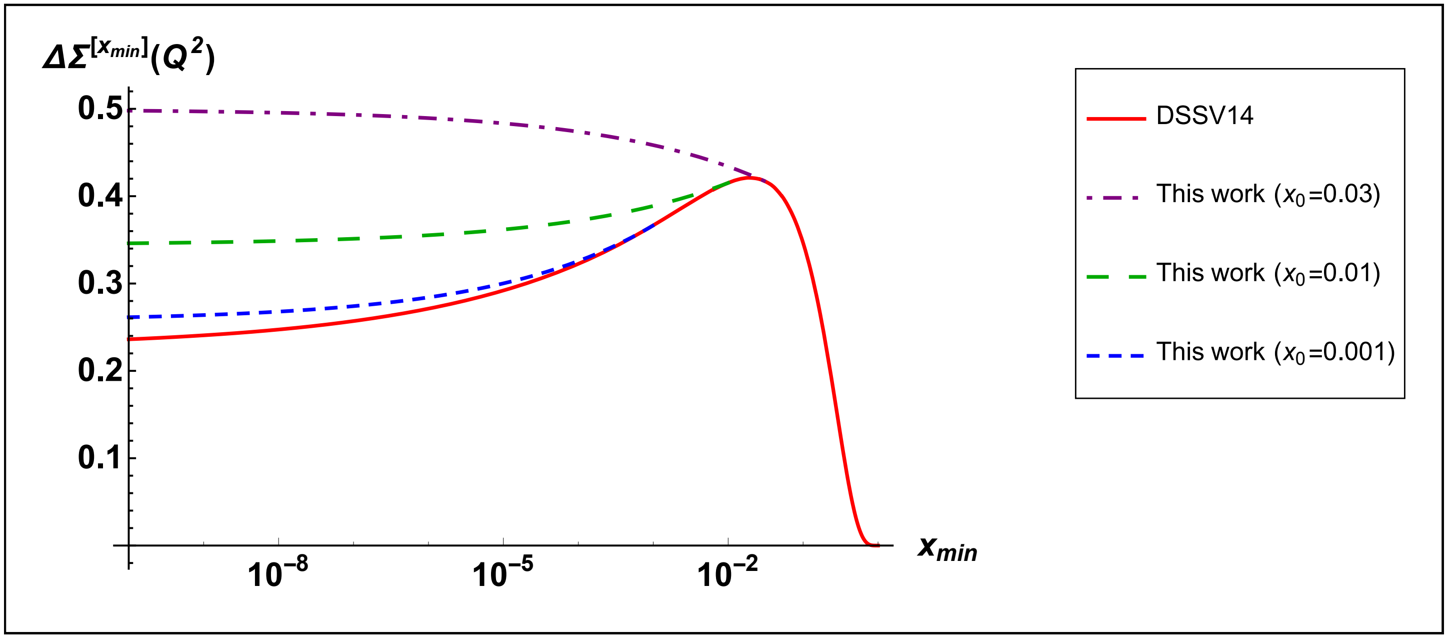
- We have attached a $\Delta\tilde{\Sigma}(x,Q^2) = N\,x^{-\alpha_h}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



"ballpark" phenomenology

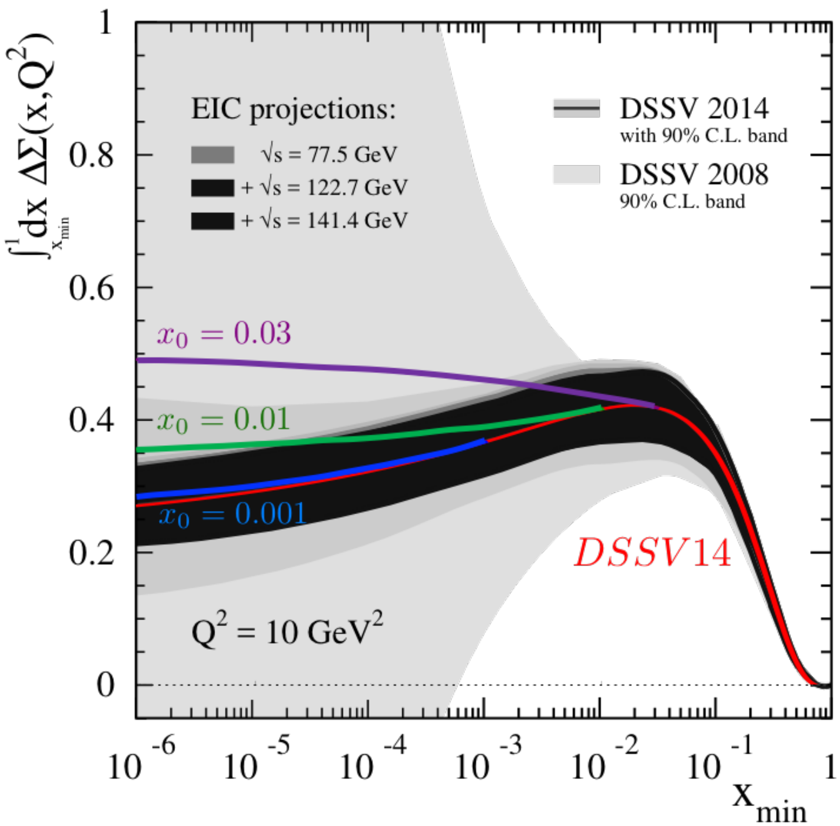
Impact of our $\Delta\Sigma$ on the proton spin

- Defining $\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta\Sigma(x, Q^2)$ we plot it for x_0=0.03, 0.01, 0.001:



- We observe a moderate to significant enhancement of quark spin.
- More detailed phenomenology is needed in the future.

Impact on proton spin



- Here we compare our results with DSSV, now including the error band.
- We observe consistency of our lower two curves with DSSV.
- Our upper curve disagrees with DSSV, but agrees with NNPDF (Nocera, Santopinto, '16).
- Better phenomenology is needed. EIC would definitely play a role.

Gluon Helicity TMDs

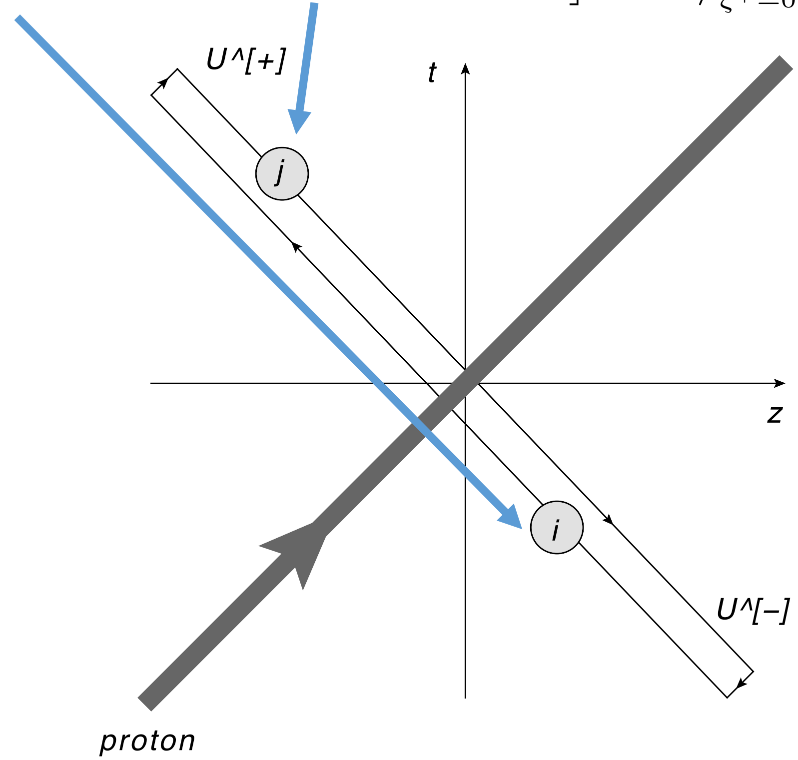
Dipole Gluon Helicity TMD

- Now let us repeat the calculation for gluon helicity TMDs.

- We start with the definition of the gluon dipole helicity TMD:

$$g_1^G(x, k_T^2) = \frac{-2i\, S_L}{x\, P^+} \int \frac{d\xi^- \, d^2\xi}{(2\pi)^3} \, e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \left\langle P, S_L | \epsilon_T^{ij} \, \mathrm{tr} \left[F^{+i}(0) \, \mathcal{U}^{[+]\dagger}[0, \xi] \, F^{+j}(\xi) \, \mathcal{U}^{[-]}[\xi, 0] \right] | P, S_L \right\rangle_{\xi^+ = 0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past Wilson line staples (hence the name `dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Dipole Gluon Helicity TMD

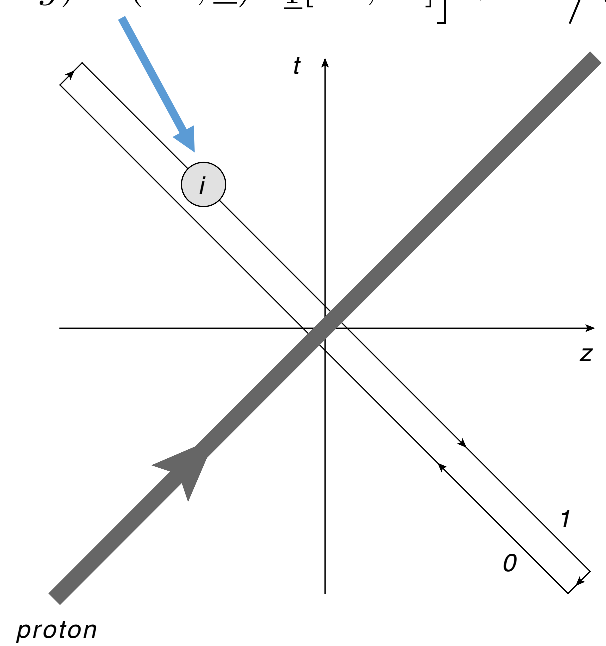
- At small x, the definition of dipole gluon helicity TMD can be massaged into

$$g_1^{G\,dip}(x, k_T^2) = \frac{8i\,N_c\,S_L}{g^2(2\pi)^3} \int d^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\, k_\perp^i\, \epsilon_T^{ij} \left[\int d^2 b_{10}\, G_{10}^j\left(zs = \frac{Q^2}{x}\right) \right]$$

- Here we obtain a new operator, which is a transverse vector (written here in A⁻=0 gauge):

$$G_{10}^i(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig)\, \tilde{A}^i(x^-, \underline{x})\, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

- Note that $k_\perp^i\, \epsilon_T^{ij}$ can be thought of as a transverse curl acting on $G_{10}^i(z)$ and not just on $\tilde{A}^i(x^-, \underline{x})$ -- different from the polarized dipole amplitude!



Dictionary

- We seem to have two operators:

- Quark helicity

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

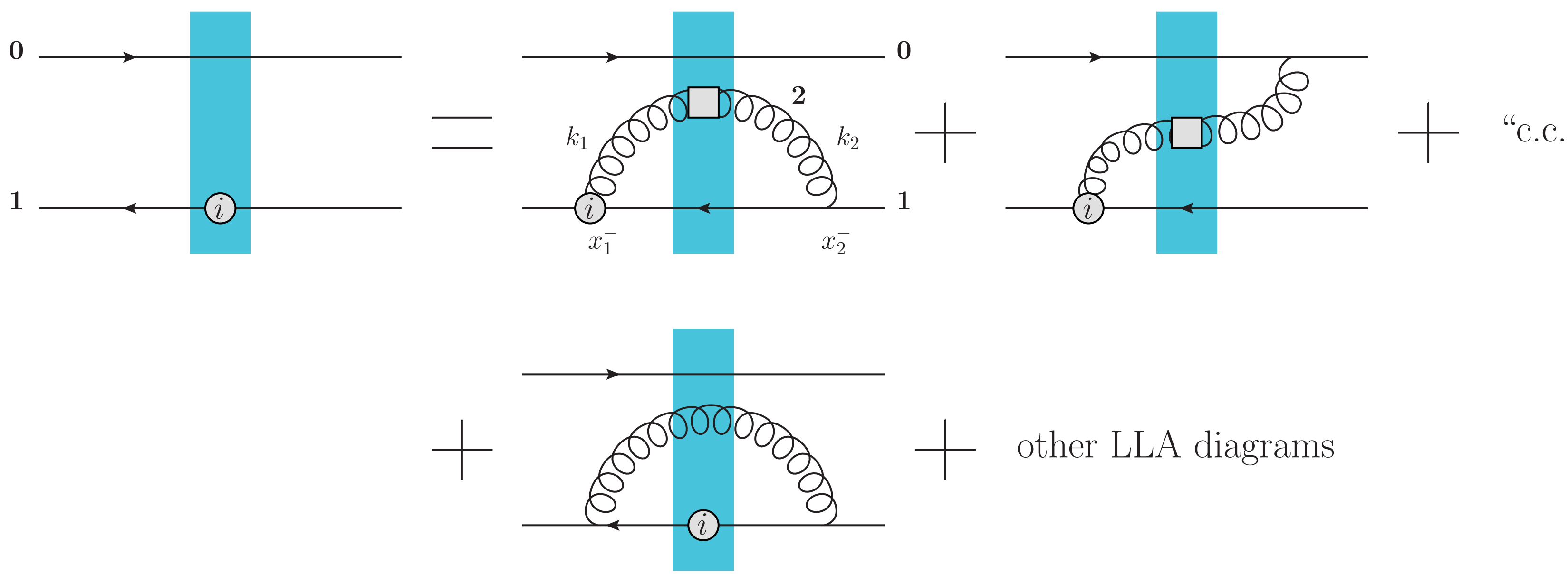
- Gluon helicity

$$G^i_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \mathrm{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \mathrm{c.c.} \right\rangle (z)$$

Gluon Helicity TMDs: Small-x Evolution

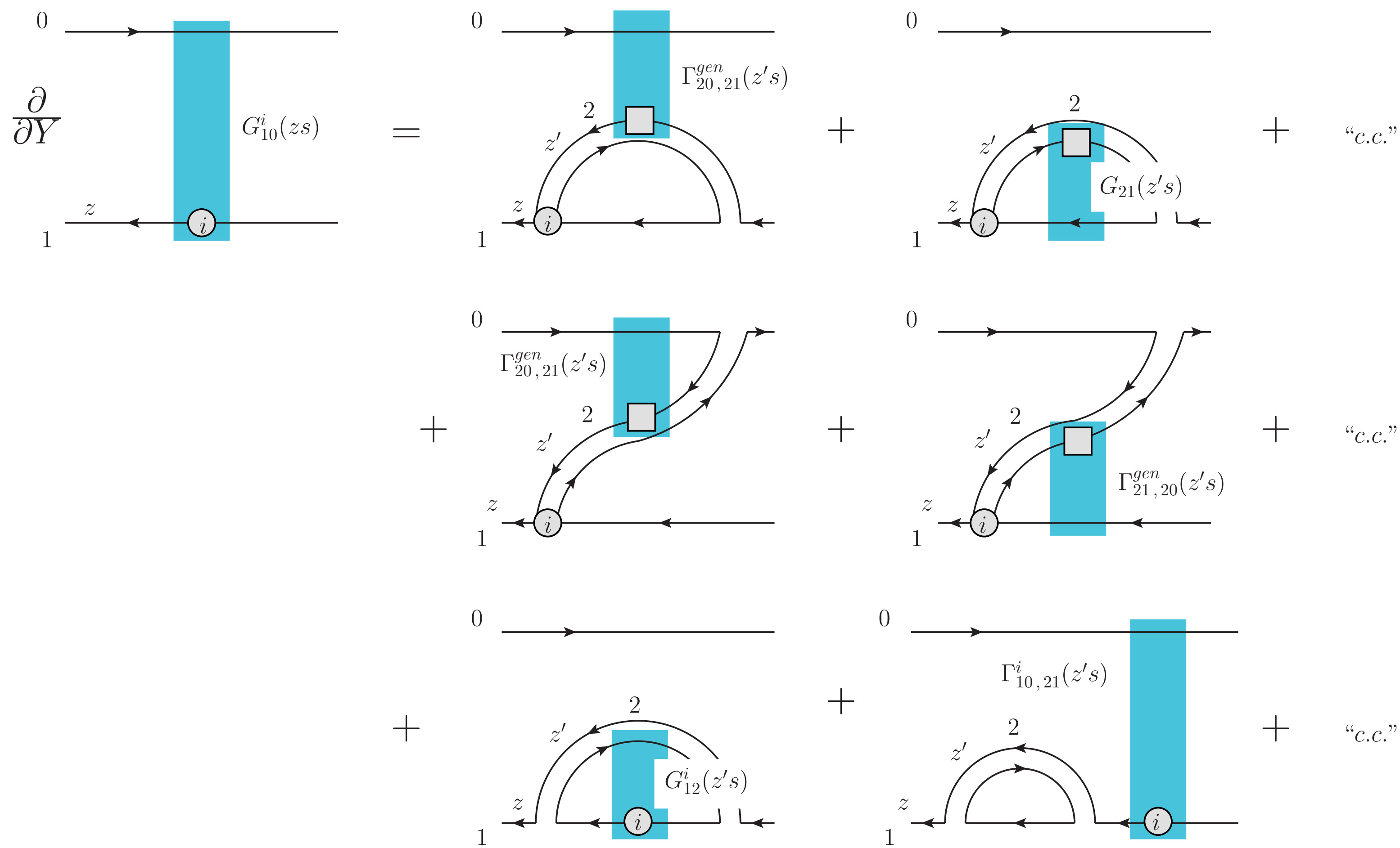
Evolution Equation

- To construct evolution equation for the operator G^i governing the gluon helicity TMD we resum similar (to the quark case) diagrams:



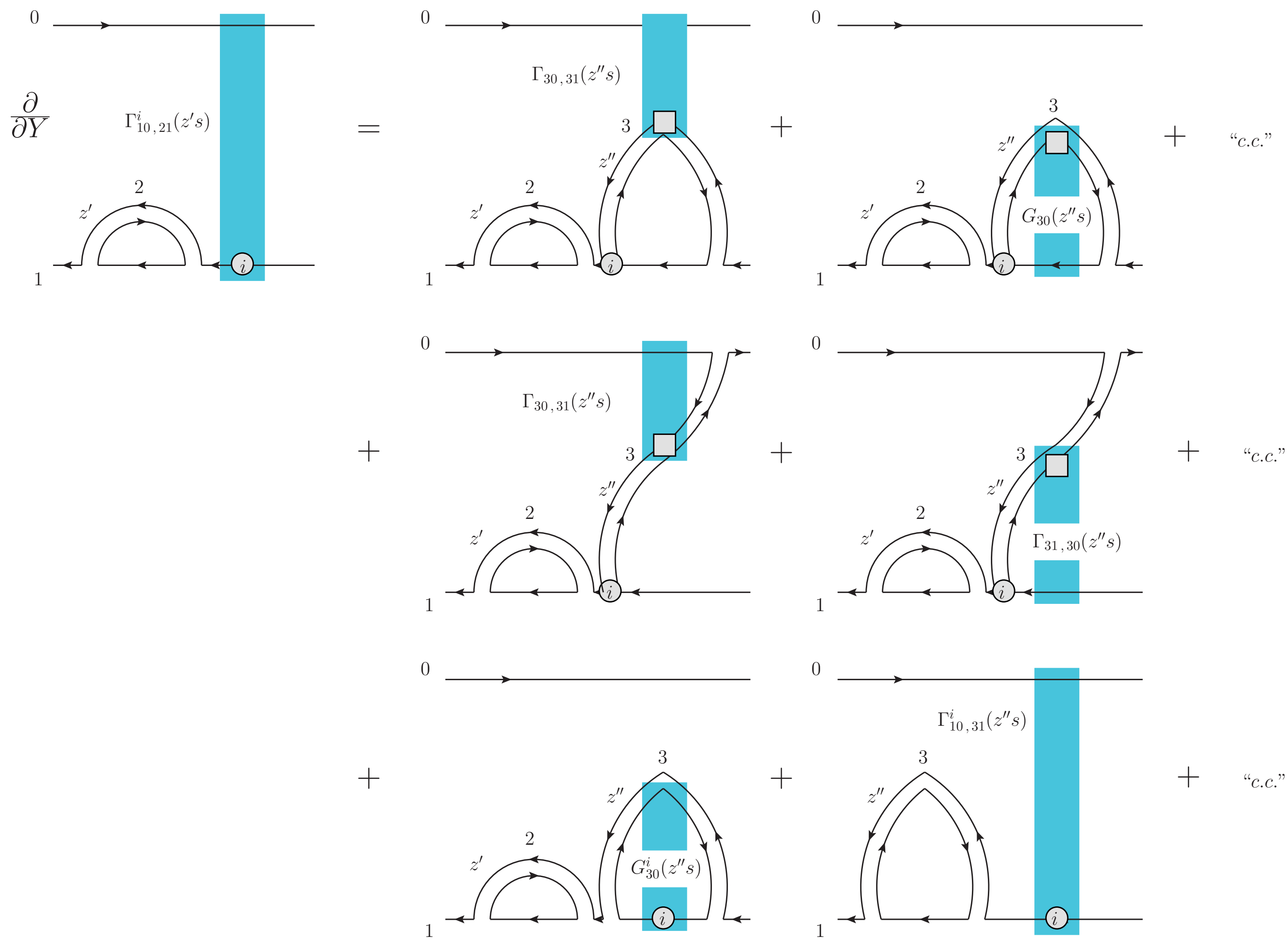
Large-N_c Evolution: Diagrams

- At large-N_c the equations are



Large-N_c Evolution: Diagrams

- and



Large-N_c Evolution: Equations

- This results in the following evolution equations:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{21})^j_\perp}{x_{21}^2} \, \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{20})^j_\perp}{x_{20}^2} \, \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's)\right]$$

$$\Gamma^i_{10\,21}(z's) = G^{i\,(0)}_{10}(z's) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{31})^j_\perp}{x_{31}^2} \, \left[\Gamma^{gen}_{30\,,\,31}(z''s) + G_{31}(z''s)\right]$$

$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z'} \frac{dz''}{z''} \int d^2 x_3 \, \ln \frac{1}{x_{31}\Lambda} \, \frac{\epsilon_T^{ij} \, (x_{30})^j_\perp}{x_{30}^2} \, \left[\Gamma^{gen}_{30\,,\,31}(z''s) + \Gamma^{gen}_{31\,,\,30}(z''s)\right]$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2\,,\,x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2} \left[G^i_{13}(z''s) - \Gamma^i_{10\,,\,31}(z''s)\right].$$

Large-N_c Evolution: Equations

- Here

$$\Gamma^{gen}_{20,21}(z's) = \theta(x_{20} - x_{21})\,\Gamma_{20,21}(z's) + \theta(x_{21} - x_{20})\,G_{20}(z's)$$

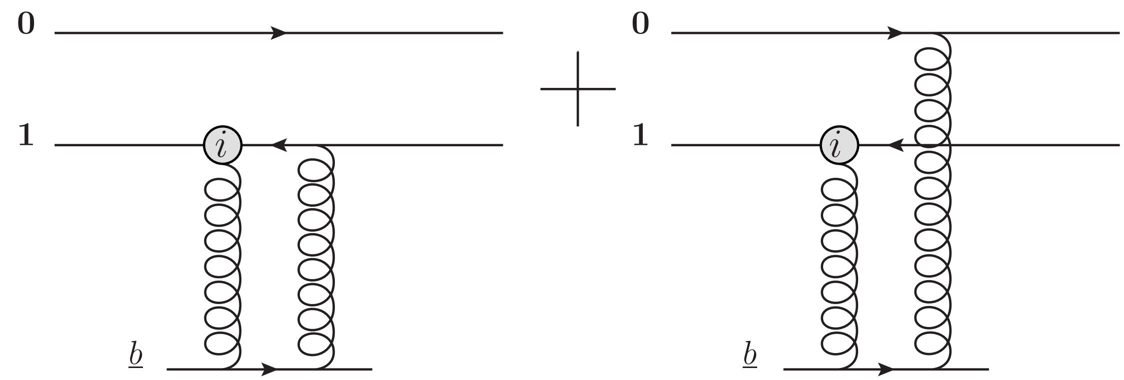
  is an object which we know from the quark helicity evolution, as the latter gives us G and Γ.

- Note that our evolution equations mix the gluon (G^i) and quark (G) small-x helicity evolution operators:

$$G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\,(x_{21})_\perp^j}{x_{21}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + G_{21}(z's)\right]$$
$$- \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \ln \frac{1}{x_{21}\Lambda} \, \frac{\epsilon_T^{ij}\,(x_{20})_\perp^j}{x_{20}^2} \left[\Gamma^{gen}_{20\,,\,21}(z's) + \Gamma^{gen}_{21\,,\,20}(z's)\right]$$
$$+ \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[G^i_{12}(z's) - \Gamma^i_{10\,,\,21}(z's)\right]$$

Initial Conditions

- Initial conditions for this evolution are given by the lowest order t-channel gluon exchanges:



$$\int d^2 b_{10}\, G_{10}^{i\,(0)}(zs) = \int d^2 b_{10}\, \Gamma_{10,21}^{i\,(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\,\pi\,\epsilon^{ij}\,x_{10}^j\,\ln\frac{1}{x_{10}\,\Lambda}$$

- Note that these initial conditions have no ln s, unlike the initial conditions for the quark evolution:

$$\int d^2 b_{10}\, G_{10}^{(0)}(zs) = \int d^2 b_{10}\, \Gamma_{10,21}^{(0)}(zs) = -\frac{\alpha_s^2 C_F}{N_c}\pi\,\ln(zs\,x_{10}^2)$$

Gluon Helicity TMDs: Small-x Asymptotics

Large-N_c Evolution Equations: Scaling

- Just like in the quark helicity evolution case, the equations simplify once we recognize the following scaling property:

$$G_2(x_{10}^2, zs) = G_2\left(\sqrt{\frac{\alpha_s\, N_c}{2\pi}}\, \ln(zs x_{10}^2)\right)$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \Gamma_2\left(\sqrt{\frac{\alpha_s\, N_c}{2\pi}}\, \ln(z's x_{10}^2), \sqrt{\frac{\alpha_s\, N_c}{2\pi}}\, \ln(z's x_{21}^2)\right)$$

- The equations become

$$G_2(\zeta) = -\frac{1}{2}\sqrt{\frac{\alpha_s\, N_c}{6\pi}}\, e^{\frac{4}{\sqrt{3}}\zeta} - \int\limits_0^{\zeta} d\xi \int\limits_0^{\xi} d\xi'\, \Gamma_2(\xi, \xi'),$$

$$\Gamma_2(\zeta, \zeta') = -\frac{1}{2}\sqrt{\frac{\alpha_s\, N_c}{6\pi}}\, e^{\frac{4}{\sqrt{3}}\zeta} - \int\limits_0^{\zeta'} d\xi \int\limits_0^{\xi} d\xi'\, \Gamma_2(\xi, \xi') - \int\limits_{\zeta'}^{\zeta} d\xi \int\limits_0^{\zeta'} d\xi'\, \Gamma_2(\xi, \xi')$$

Large-N_c Evolution Equations: Solution

- These equations can be solved in the asymptotic high-energy region using a combination of ODE solving and Laplace transform, yielding

$$G_2(\zeta \gg 1) = -\frac{1}{3}\sqrt{\frac{2\,\alpha_s\,N_c}{\pi}}\,\frac{19\sqrt{3}}{64}\,e^{\frac{13}{4\sqrt{3}}\,\zeta},$$

$$\Gamma_2(\zeta \gg 1, \zeta' \gg 1) = -\frac{1}{3}\sqrt{\frac{2\,\alpha_s\,N_c}{\pi}}\left[\frac{\sqrt{3}}{4}\,e^{\frac{4}{\sqrt{3}}\zeta - \frac{\sqrt{3}}{4}\,\zeta'} + \frac{3\sqrt{3}}{64}\,e^{\frac{4}{\sqrt{3}}\zeta' - \frac{\sqrt{3}}{4}\,\zeta}\right]$$

- The small-x gluon helicity intercept is

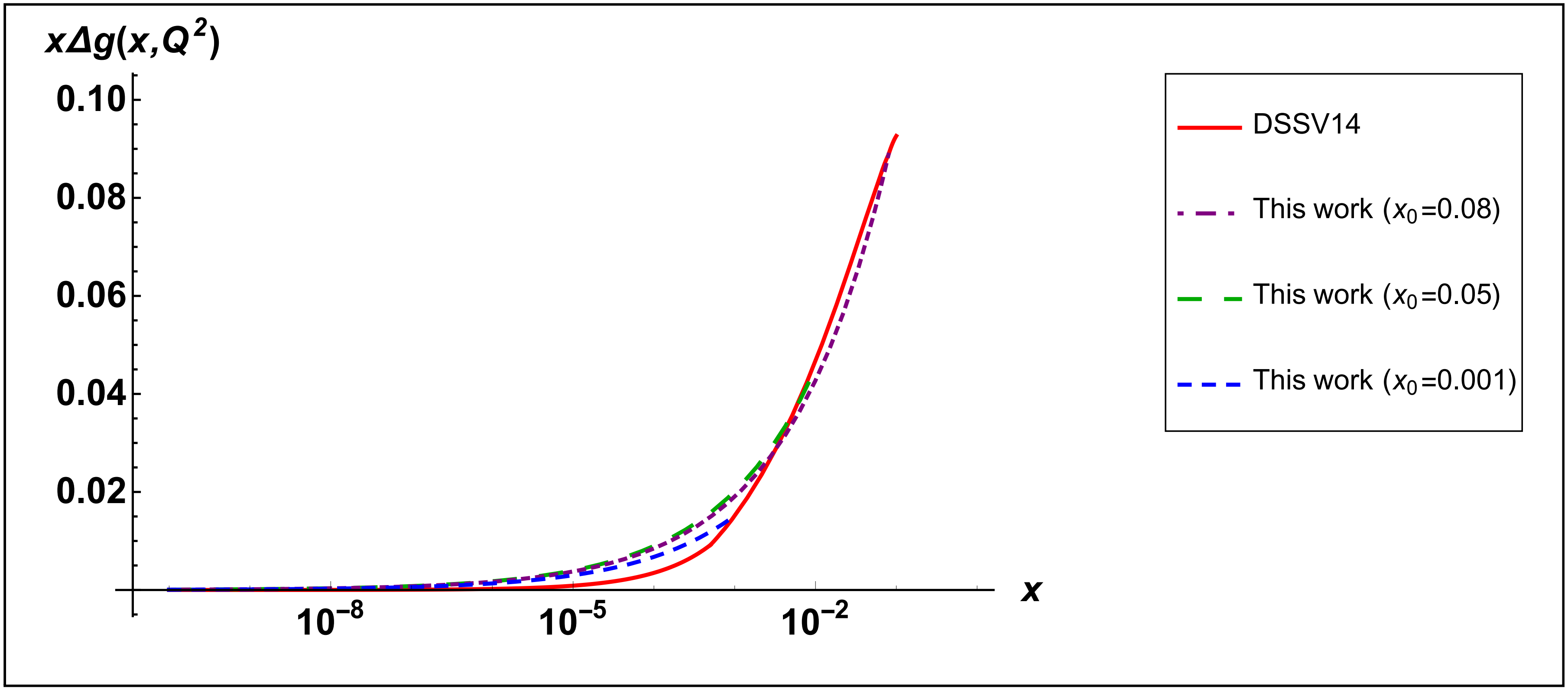
$$\alpha_h^G = \frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\,N_c}{2\pi}} \approx 1.88\sqrt{\frac{\alpha_s\,N_c}{2\pi}}$$

- We obtain the small-x asymptotics of the gluon helicity distributions:

$$\Delta G(x, Q^2) \sim g_{1L}^{G\,dip}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\frac{13}{4\sqrt{3}}\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

Impact of our ΔG on the proton spin

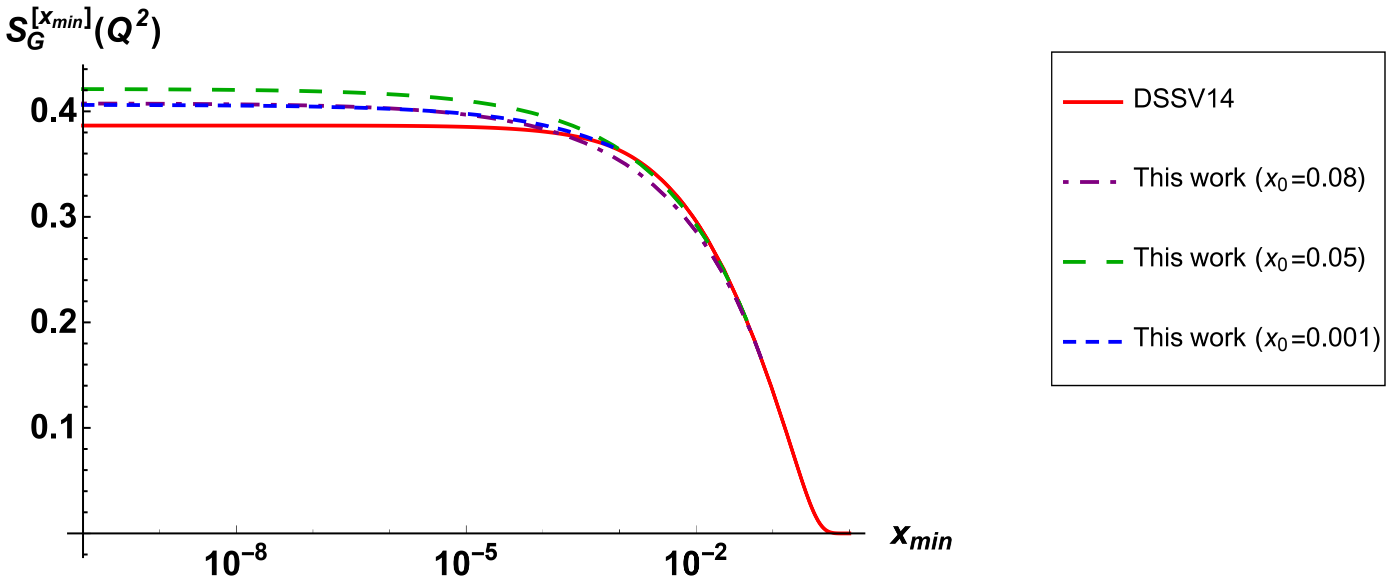
- We have attached a $\Delta\tilde{G}(x,Q^2) = N\,x^{-\alpha_h^G}$ curve to the existing hPDF's fits at some ad hoc small value of x labeled x_0 :



"ballpark"
phenomenology

Impact of our ΔG on the proton spin

- Defining $S_G^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx\, \Delta G(x, Q^2)$ we plot it for x_0=0.08, 0.05, 0.001:



- We observe a moderate enhancement of gluon spin.
- More detailed phenomenology is needed in the future.

Conclusions

- We conclude that the small-x asymptotics of gluon helicity (at large N_c) is

$$\Delta G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^G} \quad \text{with} \quad \alpha_h^G = \frac{13}{4\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 1.88 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

while the quark helicity asymptotics is

$$\Delta q(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^q} \quad \text{with} \quad \alpha_h^q = \frac{4}{\sqrt{3}} \sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 2.31 \sqrt{\frac{\alpha_s N_c}{2\pi}}$$

- Preliminary results indicate a possible enhancement of quark and gluon spin coming from small x as compared to DSSV.
- Future work may involve including running coupling and saturation corrections + solving the large-N_C&N_f equations. We can use our method to determine the small-x asymptotics of quark and gluon OAMs.
- One may use our approach to combine experiment and theory to constrain the quark and gluon spin (and OAM) at small x (in progress, a long-term goal).

INT Program on EIC Physics, Fall 2018

- **Probing Nucleons and Nuclei in High Energy Collisions** (INT-18-3)
  October 1 - November 16, 2018
  Y. Hatta, Y. Kovchegov, C. Marquet, A. Prokudin
- Institute for Nuclear Theory, Seattle, WA

- Please mark
  your calendars!

Backup Slides

Large-N_c&N_f Evolution

- The evolution equations read (in the strict DLA limit, S=1):

$$Q_{01}(z) = Q_{01}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10} - x_{21}) \, \left[G_{12}(z') + \Gamma_{02,\,21}(z') + A_{21}(z') - \bar{\Gamma}_{01,\,21}(z') \right]$$

$$+ \frac{\alpha_s N_c}{4\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10}^2 z - x_{21}^2 z') \, A_{21}(z'),$$

$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10} - x_{21}) \, \left[\Gamma_{02,\,21}(z') + 3\, G_{12}(z') \right]$$

$$- \frac{\alpha_s N_f}{4\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10}^2 z - x_{21}^2 z') \, \bar{\Gamma}_{02,\,21}(z'),$$

$$A_{01}(z) = A_{01}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10} - x_{21}) \, \left[G_{12}(z') + \Gamma_{02,\,21}(z') + A_{21}(z') - \bar{\Gamma}_{01,\,21}(z') \right]$$

$$+ \frac{\alpha_s N_c}{4\pi^2} \int\limits_{z_i}^{z} \frac{dz'}{z'} \int\limits_{\rho'^2} \frac{d^2 x_2}{x_{21}^2} \, \theta(x_{10}^2 z - x_{21}^2 z') \, A_{12}(z').$$

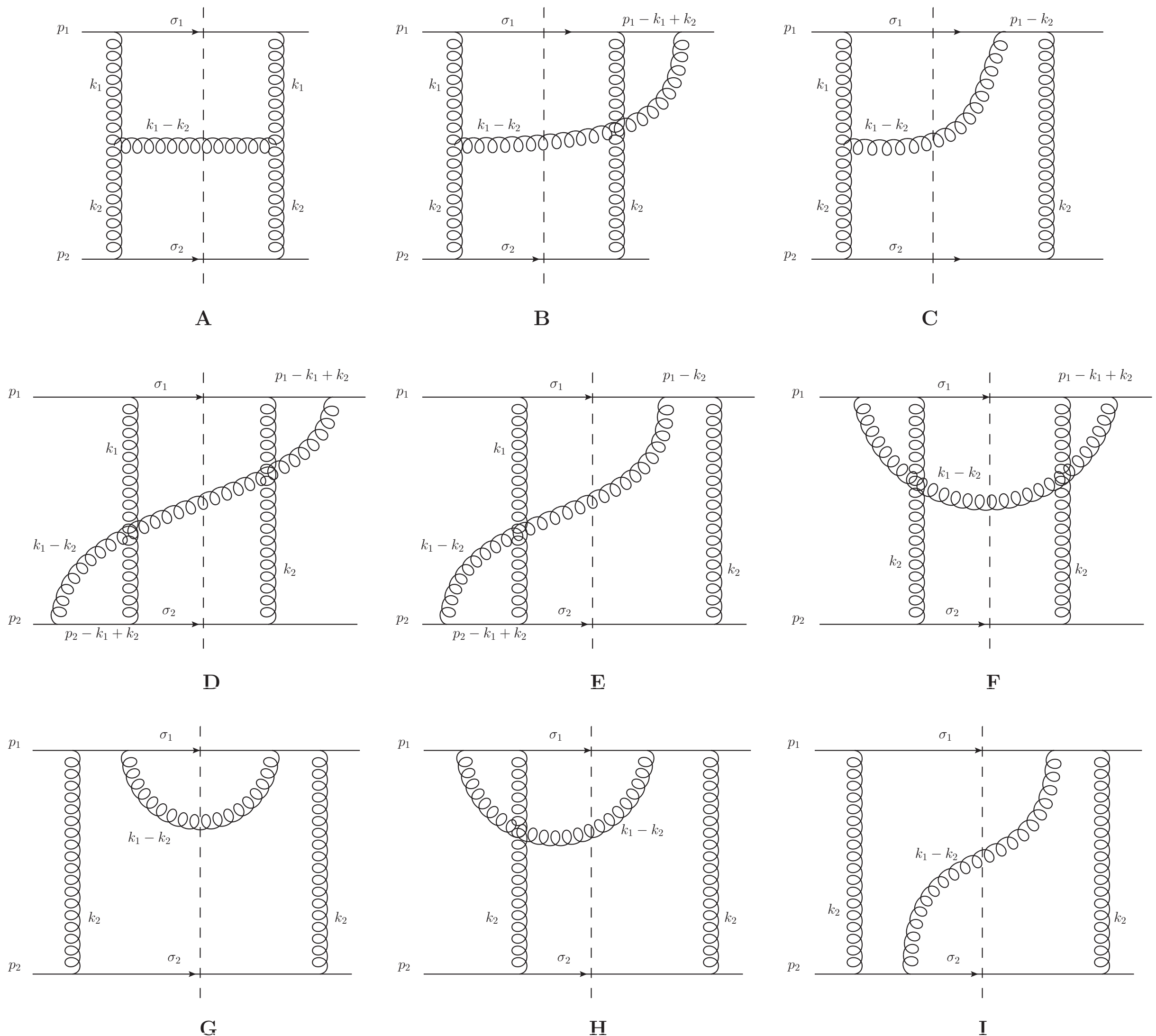
$$\Gamma_{02,\,21}(z') = \Gamma_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \, \left[\Gamma_{03,\,32}(z'') + 3\, G_{23}(z'') \right]$$

$$- \frac{\alpha_s N_f}{4\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \, \bar{\Gamma}_{03,\,32}(z'),$$

$$\bar{\Gamma}_{02,\,21}(z') = \bar{\Gamma}_{02,\,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{\min\{x_{02}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \, \left[\Gamma_{03,\,32}(z'') + G_{23}(z'') + A_{23}(z'') - \bar{\Gamma}_{02,\,32}(z'') \right]$$

$$+ \frac{\alpha_s N_c}{4\pi} \int\limits_{z_i}^{z'} \frac{dz''}{z''} \int\limits_{\rho''^2}^{x_{21}^2 z'/z''} \frac{dx_{32}^2}{x_{32}^2} \, A_{32}(z').$$

Comparison with BER

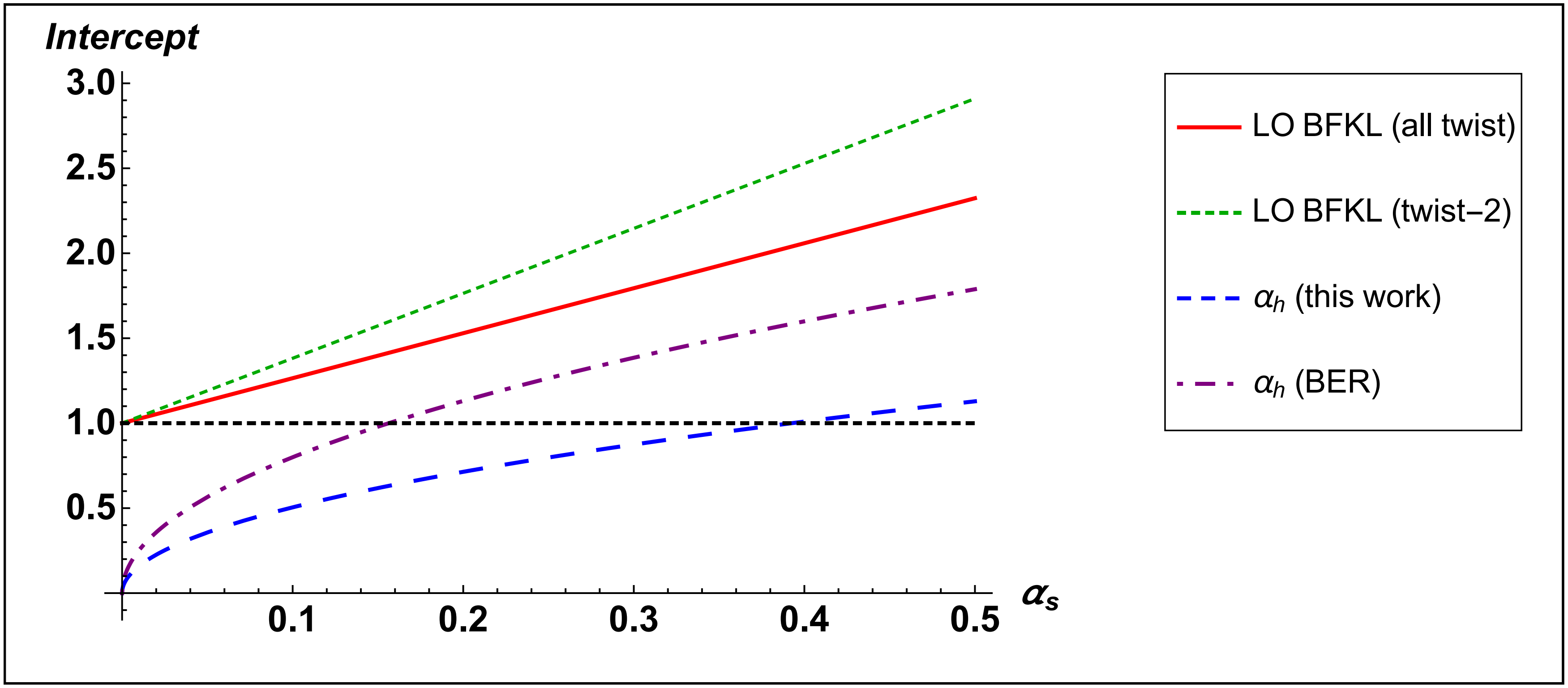


To better understand BER work, we tried calculating one (real) step of DLA helicity evolution for the qq->qq scattering.

It appears that we have identified the k_2>> k_1 (or k_1>> k_2) regime in which diagrams A, B, C, D, E, I are DLA, which was not considered by BER for B, C, … I.

Intercepts

Here we plot our (flavor-singlet) helicity intercept as a function of the coupling. We show BER result and LO BFKL (all twist and leading twist) for comparison.



Helicity Evolution at Small x
flavor non-singlet case

Yu.K., D. Pitonyak, M. Sievert, arXiv:1610.06197 [hep-ph]

Flavor Non-Singlet Observables

- In the flavor non-singlet case, all helicity observables again depend on the polarized dipole amplitude:

$$g_1^{NS}(x, Q^2) = \frac{N_c}{2\,\pi^2 \alpha_{EM}} \int\limits_{z_i}^{1} \frac{dz}{z^2(1-z)} \int dx_{01}^2 \left[\frac{1}{2} \sum_{\lambda\sigma\sigma'} |\psi^T_{\lambda\sigma\sigma'}|^2_{(x_{01}^2, z)} + \sum_{\sigma\sigma'} |\psi^L_{\sigma\sigma'}|^2_{(x_{01}^2, z)} \right] G^{NS}(x_{01}^2, z),$$

$$\Delta q^{NS}(x, Q^2) = \frac{N_c}{2\pi^3} \int\limits_{z_i}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\frac{1}{zQ^2}} \frac{dx_{01}^2}{x_{01}^2} \, G^{NS}(x_{01}^2, z),$$

$$g_{1L}^{NS}(x, k_T^2) = \frac{8\,N_c}{(2\pi)^6} \int\limits_{z_i}^{1} \frac{dz}{z} \int d^2 x_{01} \, d^2 x_{0'1} \, e^{-i \underline{k} \cdot (\underline{x}_{01} - \underline{x}_{0'1})} \, \frac{\underline{x}_{01} \cdot \underline{x}_{0'1}}{x_{01}^2 x_{0'1}^2} \, G^{NS}(x_{01}^2, z)$$

- Polarized dipole amplitude is different (difference instead of sum):

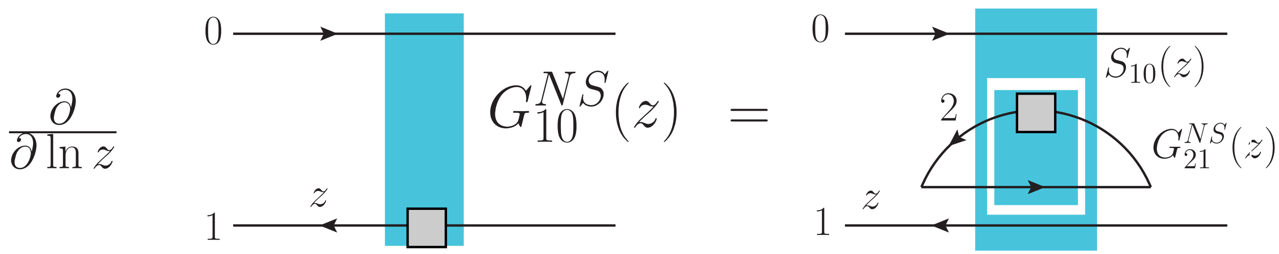
$$G_{10}^{NS}(z) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[V_{\underline{0}} V_{\underline{1}}^{pol\,\dagger} \right] - \mathrm{tr} \left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

- This is related to the definition

$$\Delta q^{NS}(x, Q^2) \equiv \Delta q^f(x, Q^2) - \Delta \bar{q}^f(x, Q^2)$$

Flavor Non-Singlet Evolution

- Evolution equations end up being much simpler in the non-singlet case:



$$G_{10}^{NS}(z) = G_{10}^{NS\,(0)}(z) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2 \frac{z}{z'}} \frac{dx_{21}^2}{x_{21}^2}\, S_{10}(z')\, G_{21}^{NS}(z')$$

- Analytical solution (in the DLA case, S=1) leads to (in agreement with Bartels et al, ‘95)

$$g_1^{NS}(x, Q^2) \sim \Delta q^{NS}(x, Q^2) \sim g_{1L}^{NS}(x, k_T^2) \sim \left(\frac{1}{x}\right)^{\alpha_h^{NS}} \approx \left(\frac{1}{x}\right)^{\sqrt{\frac{\alpha_s N_c}{\pi}}}$$

- The resulting intercept is smaller than the flavor-singlet intercept.

Dipole TMD vs dipole amplitude

- Note that the operator for the <u>dipole</u> gluon helicity TMD

$$G^i_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \tilde{A}^i(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

is different from the polarized <u>dipole</u> amplitude

$$G_{10}(z) \equiv \frac{1}{4N_c} \int\limits_{-\infty}^{\infty} dx^- \left\langle \text{tr} \left[V_{\underline{0}}[\infty, -\infty] V_{\underline{1}}[-\infty, x^-] \, (-ig) \, \underline{\nabla} \times \underline{\tilde{A}}(x^-, \underline{x}) \, V_{\underline{1}}[x^-, \infty] \right] + \text{c.c.} \right\rangle (z)$$

- We conclude that the dipole gluon helicity TMD does not depend on the polarized dipole amplitude! (Hence the ‘dipole’ name may not even be valid for such TMD.)
- This is different from the unpolarized gluon TMD case.

Large-N_c Evolution: Power Counting

- The kernel mixing G^i or Γ^i with G and Γ is LLA:

$$
\begin{aligned}
G^i_{10}(zs) = G^{i\,(0)}_{10}(zs) &+ \frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\ln\frac{1}{x_{21}\Lambda}\,\frac{\epsilon_T^{ij}\,(x_{21})^j_\perp}{x_{21}^2}\left[\Gamma^{gen}_{20\,,\,21}(z's)+G_{21}(z's)\right] \\
&- \frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\ln\frac{1}{x_{21}\Lambda}\,\frac{\epsilon_T^{ij}\,(x_{20})^j_\perp}{x_{20}^2}\left[\Gamma^{gen}_{20\,,\,21}(z's)+\Gamma^{gen}_{21\,,\,20}(z's)\right] \\
&+ \frac{\alpha_s N_c}{2\pi}\int\limits_{\frac{1}{x_{10}^2 s}}^{z}\frac{dz'}{z'}\int\limits_{\frac{1}{z's}}^{x_{10}^2}\frac{dx_{21}^2}{x_{21}^2}\left[G^i_{12}(z's)-\Gamma^i_{10\,,\,21}(z's)\right]
\end{aligned}
$$

LLA

DLA

- But, the initial conditions for G and Γ have an extra $\ln s$ as compared to G^i and Γ^i, making the two terms comparable (order-α_S^2 in $\alpha_S\, ln^2 s \sim 1$ DLA power counting).

Large-N_c Gluon Helicity Evolution Equations: Solution

- To solve the equations, first decompose the relevant object as follows:

$$\int d^2 b \, G^i_{10}(z) = x^i_{10} \, G_1(x^2_{10}, z) + \epsilon^{ij} \, x^j_{10} \, G_2(x^2_{10}, z)$$

$$\int d^2 b \, \Gamma^i_{10}(z) = x^i_{10} \, \Gamma_1(x^2_{10}, z) + \epsilon^{ij} \, x^j_{10} \, \Gamma_2(x^2_{10}, z)$$

- It turns out that only G_2 and Γ_2 contribute to evolution and to the gluon helicity TMD.

Large-N_c Evolution Equations: Solution

- Plugging in the analytic solution for the quark helicity operator, we get

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) - \frac{\alpha_s N_c}{3\pi} \frac{1}{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \left(zs x_{10}^2\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \ln \frac{1}{x_{10}\Lambda}$$

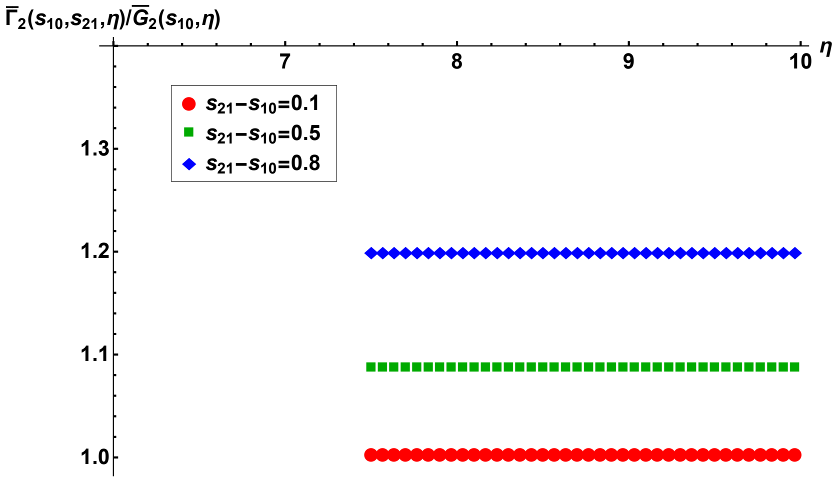
$$- \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \, \Gamma_2(x_{10}^2, x_{21}^2, z' s),$$

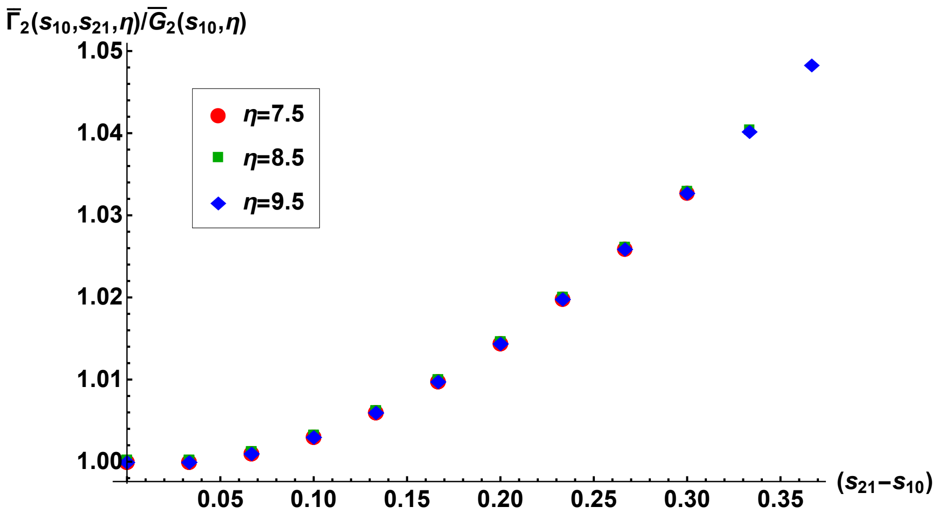
$$\Gamma_2(x_{10}^2, x_{21}^2, z' s) = G_2^{(0)}(x_{10}^2, z' s) - \frac{\alpha_s N_c}{3\pi} \frac{1}{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \left(z' s x_{10}^2\right)^{\frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}} \ln \frac{1}{x_{10}\Lambda}$$

$$- \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{x_{10}^2 s}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\left[x_{10}^2 \, , \, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{31}^2}{x_{31}^2} \, \Gamma_2(x_{10}^2, x_{31}^2, z'' s)$$

Scaling Solution Cross-Check

- One can check the scaling property $\frac{\Gamma_2}{G_2} = f(s_{21} - s_{10})$ of our analytic solution in the numerical solution of our equations:





$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{zs}{\Lambda^2}$$

$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}} \ln \frac{1}{x_{10}^2 \Lambda^2}$$

$$G_2(s_{10}, \eta) = G_2(\eta - s_{10})$$

$$\Gamma_2(s_{10}, s_{21}, \eta') = \Gamma_2(\eta' - s_{10}, \eta' - s_{21})$$