DPS @ MPI 2016

DPD sum rules in QCD

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Outline

Introduction

DPD Sum Rules

Number Sum Rule

$$\int_0^{1-x_1} \mathrm{d}x_2\, F^{j_1 j_{2,v}}(x_1, x_2) = \left(N_{j_{2,v}} + \delta_{j_1,\overline{j_2}} - \delta_{j_1,j_2}\right) f^{j_1}(x_1)$$

Momentum Sum Rule

$$\sum_{j_2} \int_0^{1-x_1} \mathrm{d}x_2\, x_2\, F^{j_1 j_2}(x_1, x_2) = (M - x_1) f^{j_1}(x_1)$$

Stirling, Gaunt 2010

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$$\sum_{j_1,j_2} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \frac{x_1 x_2}{M - x_1} F^{j_1 j_2}(x_1,x_2) = M = 1$$

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$$\int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \left(\frac{F^{j_1 j_{1,v}}(x_1,x_2)}{N_{j_{1,v}} - 1} - \frac{F^{\overline{j_1} j_{1,v}}(x_1,x_2)}{N_{j_{1,v}} + 1} \right) = N_{j_{1,v}}$$

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- consistency check: performing the following integral using either the DPD number(momentum) sum rule and the PDF momentum(number) sum rule should yield the same result

$$\sum_{j_2} \int_0^1 \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2 \, x_2 \, F^{j_1, v j_2}(x_1, x_2)$$

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$$\sum_{j_2} \int_0^{1} \mathrm{d}x_1 \int_0^{1-x_1} \mathrm{d}x_2\, x_2\, F^{j_{1,v} j_2}(x_1,x_2) = N_{j_{1,v}} - x_{j_{1,v}} \checkmark$$

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- put constraints on the DPDs and can therefore be used to refine DPD-models
- prove that these sum rules are fulfilled in QCD

Definitions

$$f^{j_1}(x_1, \boldsymbol{k}_1) = \int \frac{\mathrm{d}z_1^-}{2\pi} \mathrm{e}^{i x_1 z_1^- p^+} \int \frac{\mathrm{d}^2 \boldsymbol{z}_1^-}{(2\pi)^2} \mathrm{e}^{i \boldsymbol{z}_1 \boldsymbol{k}_1} \langle p | \bar{q}_{j_1}(-\frac{z_1}{2}) \Gamma_a \, q_{j_1}(\frac{z_1}{2}) | p \rangle$$

$$F^{j_1 j_2}(x_1, x_2, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{\Delta}) = \left[\prod_{i=1}^{2} \int \frac{\mathrm{d}z_i^-}{2\pi} \mathrm{e}^{i x_i z_i^- p^+} \int \frac{\mathrm{d}^2 \boldsymbol{z}_i^-}{(2\pi)^2} \mathrm{e}^{i \boldsymbol{z}_i \boldsymbol{k}_i} \right] \left[2p^+ \int \frac{\mathrm{d}y_1^-}{2\pi} \frac{\mathrm{d}^2 \boldsymbol{y}_1}{(2\pi)^2} \mathrm{e}^{i \boldsymbol{y}_1 \boldsymbol{\Delta}} \right]$$
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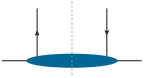
- can be interpreted in terms of Feynman diagrams, e.g.

$$f^{j_i}(x_1, \boldsymbol{k}_1) = \int \frac{\mathrm{d}z_1^-}{(2\pi)^4}$$ 

$\mathcal{O}(\alpha_s)$ example

Consider a toy-model of a meson consisting of an u-quark and $\bar{d}$-antiquark, splitting into its constituents via a pointlike coupling. For $j_1 = g$ only the following PDFs und DPDs can be realized to $\mathcal{O}(\alpha_s)$: f^g, F^{gu}, $F^{g\bar{d}}$

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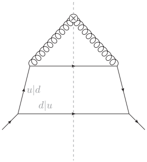
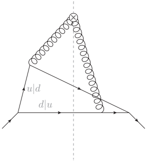


f^g



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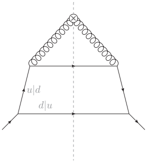
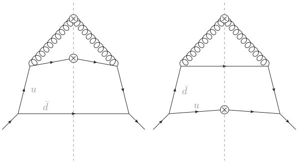
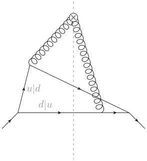
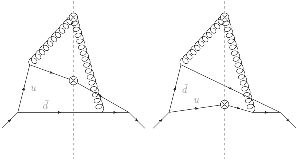




f^g

F^{gu}





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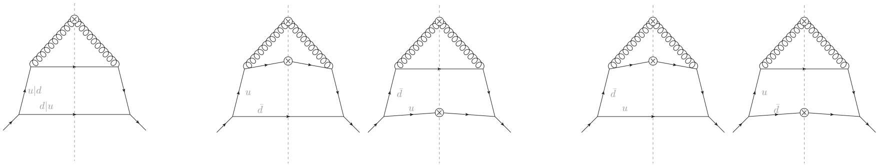

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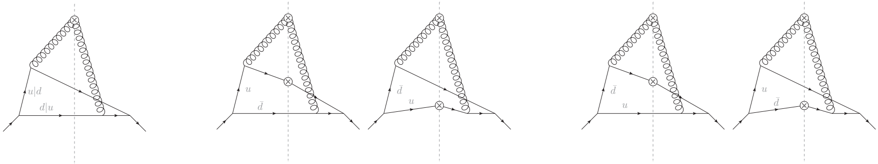

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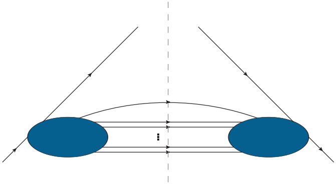

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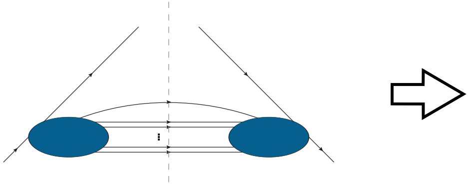

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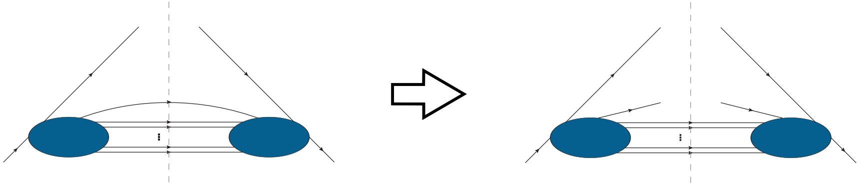

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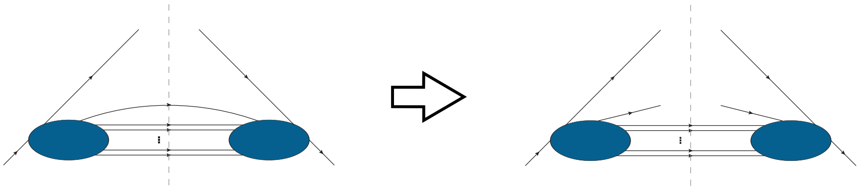

Use light-front perturbation theory to show the equivalence between PDF and DPD

Steps towards a proof for bare quantities

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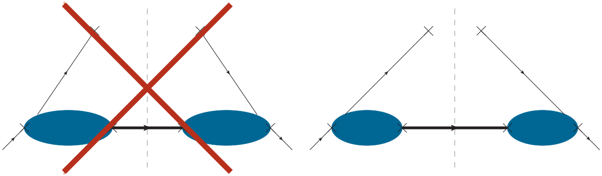
- after a sum over cuts, only such LC orderings of a PDF graph have to be considered, where there is only one state between the two hard vertices



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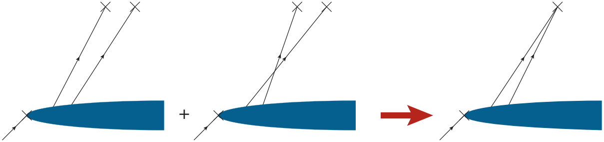

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cf. Diehl, Gaunt, Ostermeier, Plößl, Schäfer 2016

Steps towards a proof for bare quantities

PDF and DPD definitions

$$f_B^{j_1}(x_1) = \sum_t \sum_c \sum_o \left(x_1\, p^+\right)^{n_1} p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+ \right) \times \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\Big(1 - \sum_{i=1}^{M(c)} x_i\Big)$$

$$\int_0^{1-x_1} \mathrm{d}x_2\, F_B^{j_1 j_2}(x_1, x_2) = \sum_t \sum_c \sum_o \sum_l \delta_{f(l),j_2} \left(x_1 p^+\right)^{n_1} 2p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+ \right) \times \left(x_l\, p^+\right)^{n_l} \Phi^{j_1 j_2}_{DPD_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\Big(1 - \sum_{i=1}^{M(c)} x_i\Big)$$

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Main ingredient for the proof that the sum rules hold for bare quantities is to show the following relation:

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Careful ananlysis of the LCPT expressions $\Phi^{j_1\,,\,j_2}_{DPD_{t,c,o}}$ and $\Phi^{j_1}_{PDF_{t,c,o}}$ shows that this is indeed the case

Number Sum Rule

Using the relation stated above, showing the validity of the number sum rule for bare quantities reduces to showing that the following holds

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$$\sum_{l}\left(\delta_{f(l),\,j_2}-\delta_{f(l),\,\overline{j_2}}\right)=\left(N_{j_{2,v}}+\delta_{j_1,\,\overline{j_2}}-\delta_{j_1,\,j_2}\right)$$

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Using the relation stated above, showing the validity of the number sum rule for bare quantities reduces to showing that the following holds

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$$\left(N\left(j_2\right)_{t,c,o} - N\left(\bar{j}_2\right)_{t,c,o}\right) = \left(N_{j_2,v} + \delta_{j_1,\,\overline{j_2}} - \delta_{j_1,\,j_2}\right)$$

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$j_1 \neq j_2, \overline{j_2}$ $\qquad\qquad$ $\left(N_{j_{2,v}} + x\right) - x = N_{j_{2,v}}$

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$$j_1 = \overline{j_2} \qquad \left(N_{j_{2,v}} + x\right) - (x-1) = N_{j_{2,v}} + 1$$

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$$\begin{aligned}
j_1 &\neq j_2, \overline{j_2} & \left(N_{j_{2,v}} + x\right) - x &= N_{j_{2,v}} \\
j_1 &= \overline{j_2} & \left(N_{j_{2,v}} + x\right) - (x-1) &= N_{j_{2,v}} + 1 \\
j_1 &= j_2 & \left(N_{j_{2,v}} + x - 1\right) - x &= N_{j_{2,v}} - 1
\end{aligned}$$

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Using the relation stated above, showing the validity of the number sum rule for bare quantities reduces to showing that the following holds

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$$\underbrace{\begin{array}{lr} j_1 \neq j_2, \overline{j_2} & \left(N_{j_{2,v}} + x\right) - x = N_{j_{2,v}} \\ j_1 = \overline{j_2} & \left(N_{j_{2,v}} + x\right) - (x-1) = N_{j_{2,v}} + 1 \\ j_1 = j_2 & \left(N_{j_{2,v}} + x - 1\right) - x = N_{j_{2,v}} - 1 \end{array}}_{=N_{j_{2,v}} + \delta_{j_1,\,\overline{j_2}} - \delta_{j_1,\,j_2}}$$

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$$\sum_l \int \mathrm{D}_2^{N(t)}[x_i]\, \mathrm{D}_1^{N(t)}[\boldsymbol{k}_i]\; x_l\, \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\left(1-\sum_{i=1}^{M(c)} x_i\right)$$

$$= (1-x_1) \int \mathrm{D}_2^{N(t)}[x_i]\, \mathrm{D}_1^{N(t)}[\boldsymbol{k}_i]\, \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\left(1-\sum_{i=1}^{M(c)} x_i\right)$$

where

$$\int \mathrm{D}_a^b[x_i] = \prod_{i=a}^{b} \int_0^1 \mathrm{d}x_i\, p^+ \qquad \int \mathrm{D}_a^b[\boldsymbol{k}_i] = \prod_{i=a}^{b} \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}}\,,$$

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$$= (1-x_1) \int \mathrm{D}_2^{N(t)}\left[x_i\right] \mathrm{D}_1^{N(t)}\left[\boldsymbol{k}_i\right] \Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right) \delta\left(1-\sum_{i=1}^{M(c)} x_i\right)$$

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$$=(1-x_1)\int \mathrm{D}_2^{N(t)}\,[x_i]\,\mathrm{D}_1^{N(t)}\,[\boldsymbol{k}_i]\,\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\delta\left(1-\sum_{i=1}^{M(c)} x_i\right)$$

w.l.o.g. performing the x_2-integration on both sides, one finds the follwing

$$\int \mathrm{D}_3^{N(t)}\,[x_i]\,\mathrm{D}_1^{N(t)}\,[\boldsymbol{k}_i]\underbrace{\left(1-x_1-\sum_{i=3}^{M(c)} x_i+\sum_{j=3}^{M(c)} x_j\right)}_{(1-x_1)}\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\Big|_{x_2=x_{2,\,0}}$$

$$=(1-x_1)\int \mathrm{D}_3^{N(t)}\,[x_i]\,\mathrm{D}_1^{N(t)}\,[\boldsymbol{k}_i]\;\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\Big|_{x_2=x_{2,\,0}}$$

Renormalised PDFs and DPDs

PDF

$$f^{j_1}(x_1) = \sum_{i_1} \int_{x_1}^{1} \frac{\mathrm{d}z_1}{z_1} Z_{i_1 \to j_1}\left(\frac{x_1}{z_1}\right) f_B^{i_1}(z_1)$$

with renormalisation factors $Z_{i_1 \to j_1}$, which in MS-renormalisation have the following expansion in α_s

$$Z_{i_1 \to j_1}(x_1) = \delta(1-x_1)\,\delta_{i_1, j_1} + \alpha_s \frac{Z_{i_1 \to j_1;11}}{\varepsilon} + \alpha_s^2 \left(\frac{Z_{i_1 \to j_1;22}}{\varepsilon^2} + \frac{Z_{i_1 \to j_1;21}}{\varepsilon}\right) + \dots,$$

Renormalised PDFs and DPDs

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$$f^{j_1}(x_1) = Z_{i_1 \to j_1} \otimes f_B^{i_1}$$

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DPD

$$F^{j_1 j_2}(x_1, x_2) = \sum_{i_1, i_2} \int_{x_1}^{1-x_2} \frac{\mathrm{d}z_1}{z_1} \int_{x_2}^{1-z_1} \frac{\mathrm{d}z_2}{z_2} Z_{i_1 \to j_1}\left(\frac{x_1}{z_1}\right) Z_{i_2 \to j_2}\left(\frac{x_2}{z_2}\right) F_B^{i_1 i_2}(z_1, z_2)$$
$$+ \sum_{i_1} \int_{x_1+x_2}^{1} \frac{\mathrm{d}z_1}{z_1^2} Z_{i_1 \to j_1 j_2}\left(\frac{x_1}{z_1}, \frac{x_2}{z_2}\right) f_B^{i_1}(z_1)$$

with the new renormalisation factors $Z_{i_1 \to j_1 j_2}$, which are in MS-renormalisation given by

$$Z_{i_1 \to j_1 j_2} = \alpha_s \frac{Z_{i_1 \to j_1 j_2;11}}{\varepsilon} + \alpha_s^2 \left(\frac{Z_{i_1 \to j_1 j_2;22}}{\varepsilon^2} + \frac{Z_{i_1 \to j_1 j_2;21}}{\varepsilon^2} \right) + \ldots,$$

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Finally we define a inverse PDF renormalisation factor $Z^{-1}_{i'_1 \to i_1}$, obeying

inverse renormalisation factor

$$\sum_{i_1} \int_{x_1}^{1} \frac{\mathrm{d}u_1}{u_1} Z^{-1}_{i'_1, i_1}\left(\frac{x_1}{u_1}\right) Z_{i_1, j_1}(x_1) = \delta_{i'_1, j_1} \delta(1 - x_1) \ .$$

Renormalised PDFs and DPDs

PDF

$$f^{j_1}(x_1) = Z_{i_1 \to j_1} \otimes f_B^{i_1}$$

DPD

$$F^{j_1 j_2}(x_1, x_2) = Z_{i_1 \to j_1} \otimes Z_{i_2 \to j_2} \otimes F_B^{i_1 i_2} + Z_{i_1 \to j_1 j_2} \otimes f_B^{i_1}$$

inverse renormalisation factor

$$Z^{-1}_{i_1', i_1} \otimes Z_{i_1, j_1} = \delta_{i_1', j_1} \delta(1 - x_1)$$

$$f_B^{i_1} = Z^{-1}_{i_1', i_1} \otimes f^{i_1'}$$

Number Sum Rule

Subtracting the rhs of the number sum rule from the lhs and using the definitions introduced before, we find

$$\int_0^{1-x_1} \mathrm{d}x_2 F^{j_1 j_{2,v}}(x_1,x_2) - \left(N_{j_{2v}} + \delta_{j_1,\overline{j_2}} - \delta_{j_1,j_2}\right) f^{j_1}(x_1) = \sum_{i_1'} \int_{x_1}^{1} \frac{\mathrm{d}u_1}{u_1} f^{i_1'}(u_1)\, R'(x_1,u_1)$$

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where $R'(x_1,u_1)$ is given by

$$R'(x_1,u_1) =$$

$$\sum_{i_1} \int_{x_1}^{u_1} \frac{\mathrm{d}z_1}{z_1} Z^{-1}_{i_1' \to i_1}\left(\frac{x_1}{u_1}\right) \Bigg[\left(Z_{i_1 \to j_1}\left(\frac{x_1}{z_1}\right) - \delta\left(1-\frac{x_1}{z_1}\right) \delta_{i_1,j_1} \right) \left(\delta_{i_1,\overline{j_2}} - \delta_{i_1,j_2} - \delta_{j_1,\overline{j_2}} + \delta_{j_1,j_2} \right)$$

$$+ \int_0^{1-\frac{x_1}{z_1}} \mathrm{d}u_2 \left(Z_{i_1 \to j_1 j_2}\left(\frac{x_1}{z_1}, u_2\right) - Z_{i_1 \to j_1 \overline{j_2}}\left(\frac{x_1}{z_1}, u_2\right) \right) \Bigg] .$$

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As we now know that $R' = 0$ we can derive the following relation between the renormalisation factors for the inhomogeneous term and the regular PDF renormalisation factors

Number Sum Rule for renormalisation facotrs

$$\int_0^{1-x_1} \mathrm{d}x_2 \left(Z_{i_1\to j_1 j_2}(x_1,x_2) - Z_{i_1\to j_1 \overline{j_2}}(x_1,x_2)\right) = \left(\delta_{i_1,j_2} - \delta_{i_1,\overline{j_2}} + \delta_{j_1,\overline{j_2}} - \delta_{j_1,j_2}\right) Z_{i_1\to j_1}(x_1)$$

Momentum Sum Rule

Repeating the same for the momentum sum rule one finds

$$\sum_{j_2} \int_0^{1-x1} \mathrm{d}x_2\, x_2 F^{j_1 j_2}(x_1, x_2) - (1 - x_1)\, f^{j_1}(x_1) = \sum_{i_1'} \int_{x_1}^{1} \frac{\mathrm{d}u_1}{u_1} f^{i_1'}(u_1)\, R(x_1, u_1)$$

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where $R(x_1, u_1)$ is given by

$$R(x_1,\, u_1) = \sum_{i_1} \int_{x_1}^{u_1} \frac{\mathrm{d}z_1}{z_1} Z^{-1}_{i'_1 \to i_1}\left(\frac{x_1}{u_1}\right) \left[\left(Z_{i_1 \to j_1}\left(\frac{x_1}{z_1}\right) - \delta\left(1 - \frac{x_1}{z_1}\right) \delta_{i_1, j_1} \right) (x_1 - z_1) \right.$$
$$\left. + z_1 \sum_{j_2} \int_0^{1-\frac{x_1}{z_1}} \mathrm{d}u_2\, u_2\, Z_{i_1 \to j_1 j_2}\left(\frac{x_1}{z_1}, u_2\right) \right]$$

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Repeating the same for the momentum sum rule one finds

$$\sum_{j_2} \int_0^{1-x1} \mathrm{d}x_2 \, x_2 F^{j_1 j_2}(x_1, x_2) - (1 - x_1) f^{j_1}(x_1) = \sum_{i_1'} \int_{x_1}^{1} \frac{\mathrm{d}u_1}{u_1} f^{i_1'}(u_1) R(x_1, u_1)$$

Using the same reasoning as in the case of the number sum rule one can thus conclude, that also the momentum sum rule holds for renormalised quantities. The constraint, that $R = 0$ yields the follwing relation between $Z_{i_1 \to j_1 j_2}$ and $Z_{i_1 \to j_1}$

Momentum Sum Rule for renormalisation factors

$$\sum_{j_2} \int_0^{1-x_1} \mathrm{d}x_2 \, x_2 \, Z_{i_1 \to j_1 j_2}(x_1, x_2) = (1 - x_1) Z_{i_1 \to j_1}(x_1)$$

QCD evolution of PDFs and DPDs

DGLAP Equation

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\mu^2\right)} f^{j_1}\left(x_1\right) = \sum_{i_1} \int_{x_1}^{1} \frac{\mathrm{d}z_1}{z_1} P_{i_1 \to j_1}\left(\frac{x_1}{z_1}\right) f^{i_1}\left(z_1\right)$$

where $P_{i_1 \to j_1}$ are the well known DGLAP splitting kernels.

QCD evolution of PDFs and DPDs

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$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\mu^2\right)} f^{j_1} = P_{i_1 \to j_1} \otimes f^{i_1}$$

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proposed dDGLAP equation

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\mu^2\right)} F^{j_1 j_2}\left(x_1, x_2\right) = \sum_{i_1} \int\limits_{x_1}^{1-x_2} \frac{\mathrm{d}z_1}{z_1} P_{i_1\to j_1}\left(\frac{x_1}{z_1}\right) F^{i_1 j_2}\left(z_1, x_2\right)$$
$$+ \sum_{i_2} \int\limits_{x_2}^{1-x_1} \frac{\mathrm{d}z_2}{z_2} P_{i_2\to j_2}\left(\frac{x_2}{z_2}\right) F^{j_1 i_2}\left(x_1, z_2\right) + \sum_{i_1} \int\limits_{x_1+x_2}^{1} \frac{\mathrm{d}z_1}{z_1^2} P_{i_1\to j_1 j_2}\left(\frac{x_1}{z_1}, \frac{x_2}{z_1}\right) f^{i_1}\left(z_1\right)$$

where the $P_{i_1\to j_1 j_2}$ are $1 \to 2$ splitting kernels about which not much is known a priori

QCD evolution of PDFs and DPDs

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$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\mu^2\right)} F^{j_1 j_2} = P_{i_1 \to j_1} \otimes F^{i_1 j_2} + P_{i_2 \to j_2} \otimes F^{j_1 i_2} + P_{i_1 \to j_1 j_2} \otimes f^{i_1}$$

QCD evolution of PDFs and DPDs

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- the form of the dDGLAP equation is a generalization of LO and NLO results

Kirschner 1979
Ceccopieri 2011,2014

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- the form of the dDGLAP equation is a generalization of LO and NLO results

  Kirschner 1979
  Ceccopieri 2011,2014
- by comparing our proposed form of the dDGLAP equation to the explicit μ-dependence of the renormalised DPD and using the relations obtained from the validity of the sum rules for renormalised quantities we were able to derive analogous sum rules for the $1 \to 2$ splitting kernels

Consistency Checks

- comparing the μ-dependence of the renormalised DPD to the dDGLAP-equation one finds the following relation

dDGLAP equation for renormalisation factors

$$\frac{\mathrm{d}}{\mathrm{d}\log\left(\mu^2\right)} Z_{i_1'\to j_1 j_2}\left(x_1,x_2\right) = \sum_{i_1} \int_{x_1}^{1-x_2} \frac{\mathrm{d}z_1}{z_1} P_{i_1\to j_1}\left(\frac{x_1}{z_1}\right) Z_{i_1'\to i_1 j_2}\left(z_1,x_2\right)$$

$$+ \sum_{i_2} \int_{x_2}^{1-x_1} \frac{\mathrm{d}z_2}{z_2} P_{i_2\to j_2}\left(\frac{x_2}{z_2}\right) Z_{i_1'\to j_1 i_2}(x_1,z_2) + \sum_{i_1} \int_{x_1+x_2}^{1} \frac{\mathrm{d}z_1}{z_1^2} P_{i_1\to j_1 j_2}\left(\frac{x_1}{z_1},\frac{x_2}{z_1}\right) Z_{i_1'\to i_1}(z_1)$$

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- exactly the same structure as the dDGLAP equation, just like in the case of the regular PDF renormalisation factors

Consistency Checks

dDGLAP equation for renormalisation factors

$$\frac{\mathrm{d}}{\mathrm{d}\log(\mu^2)} Z_{i_1' \to j_1 j_2} = P_{i_1 \to j_1} \otimes Z_{i_1' \to i_1 j_2} + P_{i_2 \to j_2} \otimes Z_{i_1' \to j_1 i_2} + P_{i_1 \to j_1 j_2} \otimes Z_{i_1' \to i_1}$$

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In combination with the sum rules for the $1 \to 2$ renormalisation factors, this allows to obtain analogous number and momentum sum rules for the new $1 \to 2$ splitting kernels

$$\int_0^{1-x_1} \mathrm{d}x_2 \left(Z_{i_1 \to j_1 j_2}(x_1, x_2) - Z_{i_1 \to j_1 \overline{j_2}}(x_1, x_2)\right) = \left(\delta_{i_1, j_2} - \delta_{i_1, \overline{j_2}} + \delta_{j_1, \overline{j_2}} - \delta_{j_1, j_2}\right) Z_{i_1 \to j_1}(x_1)\, b$$

$$\sum_{j_2} \int_0^{1-x_1} \mathrm{d}x_2\, x_2\, Z_{i_1 \to j_1 j_2}(x_1, x_2) = (1 - x_1) Z_{i_1 \to j_1}(x_1)$$

Consistency Checks

dDGLAP equation for renormalisation factors

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Number Sum Rule for $1\to 2$ splitting kernels

$$\int_0^{1-x_1} \mathrm{d}x_2 \left(P_{i_1\to j_1 j_2}(x_1,x_2) - P_{i_1\to j_1\overline{j_2}}(x_1,x_2)\right) = \left(\delta_{i_1,j_2} - \delta_{i_1,\overline{j_2}} + \delta_{j_1,\overline{j_2}} - \delta_{j_1,j_2}\right) P_{i_1\to j_1}(x_1)$$

Momentum Sum Rule for $1\to 2$ splitting kernels

$$\sum_{j_2}\int_0^{1-x_1} \mathrm{d}x_2\, x_2\, P_{i_1\to j_1 j_2}(x_1,x_2) = (1-x_1) P_{i_1\to j_1}(x_1)$$

Consistency Checks

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- can be used to show stability of the DPD sum rules under QCD evolution

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- can be used to show stability of the DPD sum rules under QCD evolution
- as it should already be clear after the proof that the sum rules hold for renormalised quantities, that they are also stable under evolution, this acts as a consistency check for our proposed dDGLAP-equation

Summary

- we showed the validity of the DPD sum rules for bare quantities using a diagramatic approach and LCPT

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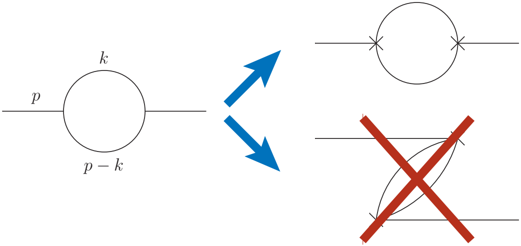
- we then discussed renormalization and showed that the sum rules are also valid for renormalised quantities
- in doing so we derived number and momentum sum rules for the $1 \to 2$ renormalisation factors
- finally we considered QCD evolution and generalized the dDGLAP-equation to higher orders
- this allowed us to derive number and momentum sum rules for the $1 \to 2$ splitting kernels
- as a consistency check we showed that with our proposed dDGLAP-equation the sum rules are preserved under evolution

LCPT I: Motivation

As an example consider a quark loop in ϕ^3 theory:



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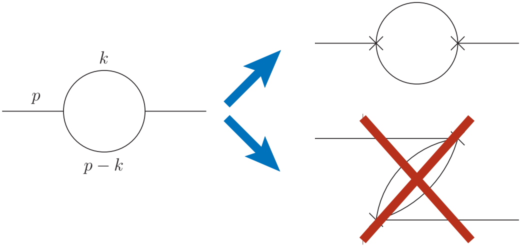

In covariant PT the loop is given by

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon}$$

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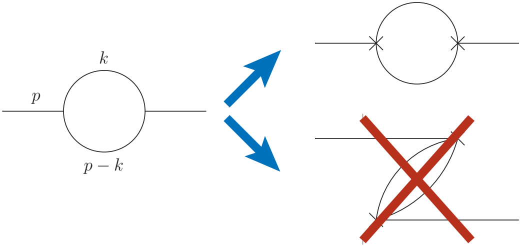

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Performing the k^- integration using Cauchy's theorem one finds

$$\int_0^{p^+} \frac{\mathrm{d}k^+}{2\pi} \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}}{(2\pi)^{D-2}} \frac{1}{(2k^+)(2(p^+ - k^+))} \frac{1}{p^- - \frac{\boldsymbol{k}^2 + m^2}{2k^+} - \frac{(\boldsymbol{p}-\boldsymbol{k})^2 + m^2}{2(p^+ - k^+)} + i\epsilon}$$

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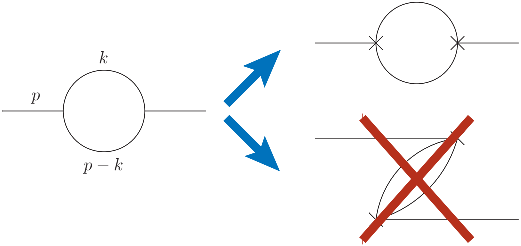

Generally the denominator for a state ζ_i between two vertices x_i and x_{i+1} is given by:

$$\frac{1}{P_i^- - \sum_{l\in i} k^-_{l,\,\text{on-shell}} + i\epsilon}$$

where P_i is the sum of all external momenta entering the graph before vertex i and the sum is over the on-shell minus momenta of all lines in the state

LCPT II: Rules

- Starting from a given Feynman diagram one has to consider all possible x^+-orderings of the vertices. In order to visualise these orderings one uses that x^+ increases from left to right on the lhs of the cut while it increases from right to left on the rhs of the cut.
- Coupling constants and vertex factors are the same as in covariant PT.
- Plus and transversal momenta, k_l^+ und $\boldsymbol{k}_l$, of a line l are conserved at the vertices
- Each line l in a graph comes with a factor $\frac{1}{2k_l^+}$ and a Heaviside function $\Theta(k_l^+)$, corresponding to propagation from lower to higher x^+
- For each loop theres an integral over plus and transversal components of the loop momentum ℓ:

$$\int \frac{\mathrm{d}\ell^+ \mathrm{d}^{d-2}\boldsymbol{\ell}}{(2\pi)^{d-1}}$$

- For each state ζ_i between two vertices x_i^+ und x_{i+1}^+ one gets the aforementioned factor

$$\frac{1}{P_i^- - \sum_{l\in i} k^-_{l,\,\mathrm{on-shell}} + i\epsilon}$$

LCPT III: PDF and DPD Definitions in LCPT

PDF

$$f_B^{j_1}(x_1) = \sum_t \sum_c \sum_o \left(k_1^+\right)^{n_1} \int \frac{\mathrm{d}k_1^- \mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^D} \Big(\prod_{i=2}^{M(c)} \frac{\mathrm{d}k_i^+ \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} \Big) \Big(\prod_{i=M(c)+1}^{N(t)} \frac{\mathrm{d}k_i^+ \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} \Big)$$

$$\times \Phi^{j_1}_{PDF_{t,c,o}}\Big(\{k^+\}, \{\boldsymbol{k}\}\Big) 2\pi\delta\Big(p^- - k^- - \sum_{i=2}^{N} k^-_{i\,,\mathrm{on-shell}}\Big) \delta\Big(p^+ - \sum_{i=1}^{N} k_i^+\Big)$$

where $n_1 = 1$ if parton 1 is a gluon or a scalar quark, while for Dirac quarks one has $n_1 = 0$.

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DPD

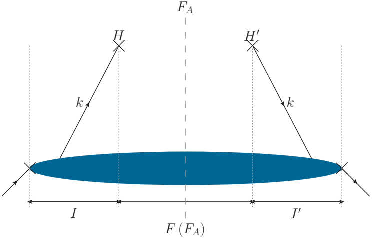
$$F_B^{j_1 j_2}(x_1, x_2) = \sum_t \sum_c \sum_o \sum_l \delta_{f(l), j_2} \left(k_1^+\right)^{n_1} \left(k_2^+\right)^{n_2} 2p^+ (2\pi)^{D-1}$$

$$\times \int \frac{\mathrm{d}k_1^- \mathrm{d}k_l^- \mathrm{d}\Delta^- \mathrm{d}^{D-2}\boldsymbol{k}_1 \mathrm{d}^{D-2}\boldsymbol{k}_l}{(2\pi)^{3D}} \Big(\prod_{i=2,\, i\neq l}^{M(c)} \frac{\mathrm{d}k_i^+ \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} \Big) \Big(\prod_{i=M(c)+1}^{N(t)} \frac{\mathrm{d}k_i^+ \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} \Big)$$

$$\times\, \Phi^{j_1 j_2}_{DPD_{t,c,o}}\left(\{k^+\}, \{\boldsymbol{k}\}\right) 2\pi\delta\Big(p^- - k_1^- - k_l^- - \sum_{i=2,\, i\neq l}^{M(c)} k^-_{i\,,\mathrm{on-shell}}\Big) \delta\Big(p^+ - \sum_{i=1}^{M(c)} k_i^+\Big)$$

LCPT IV: contributing x^+ orderings for PDFs

Consider an arbitrary LCPT PDF graph



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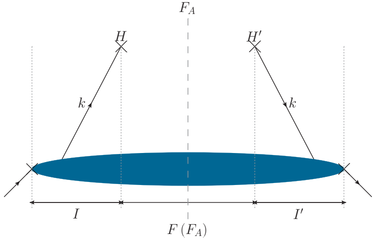

This can be decomposed as

$$\Phi_{PDF} = I\, F(F_A)\, I'$$

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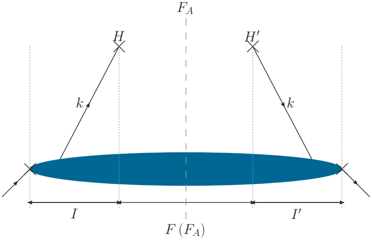

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where

$$I = \prod_{\substack{\text{states } \zeta \\ \zeta < H}} \frac{1}{p^- - \sum_{l\in\zeta} k^-_{l,\,\text{o.s.}} + i\epsilon} \qquad I' = \prod_{\substack{\text{states } \zeta \\ \zeta < H'}} \frac{1}{p^- - \sum_{l\in\zeta} k^-_{l,\,\text{o.s.}} - i\epsilon}$$

$$F(F_A) = \prod_{\substack{\text{states } \zeta \\ H<\zeta<F_A}} \frac{1}{p^- - k^- - \sum_{l\in\zeta} k^-_{l,\,\text{o.s.}} + i\epsilon} \prod_{\substack{\text{states } \zeta \\ H'<\zeta<F_A}} \frac{1}{p^- - k^- - \sum_{l\in\zeta} k^-_{l,\,\text{o.s.}} - i\epsilon}$$
$$\times\, 2\pi\delta\left(p^- - k^- - \sum_{l\in F_A} k^-_{l,\,\text{o.s.}}\right)$$

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Assuming that there are N distinct states between H and H' there are thus also N possible choices for the final state cut F_A. Summing $F(F_A)$ over all cuts one finds the following

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where

$$D_f = \sum_{l \in f} k^-_{l,\,\mathrm{on-shell}}\,,$$

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rewriting the on-shell δ function as

$$2\pi\,\delta(x) = i\left[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right]$$

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For $N \geq 2$ this expression vanishes after integration over k^- while for $N = 1$ the on-shell δ function is reproduced.

LCPT IV: contributing x^+ orderings for PDFs

Assuming that there are N distinct states between H and H' there are thus also N possible choices for the final state cut F_A. Summing $F(F_A)$ over all cuts one finds the following

$$\sum_{F_A} F(F_A) = \sum_{c=1}^{N} \left[\prod_{f=1}^{c-1} \frac{1}{p^- - k^- - D_f + i\epsilon} 2\pi\delta\left(p^- - k^- - D_c\right) \prod_{f=c+1}^{N} \frac{1}{p^- - k^- - D_f - i\epsilon} \right]$$

rewriting the on-shell δ function as

$$2\pi\,\delta(x) = i\left[\frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}\right]$$

the above equation becomes

$$\sum_{F_A} F(F_A) = i\left[\prod_{f=1}^{N} \frac{1}{p^- - k^- - D_f + i\epsilon} - \prod_{f=1}^{N} \frac{1}{p^- - k^- - D_f - i\epsilon}\right]$$

For $N \geq 2$ this expression vanishes after integration over k^- while for $N = 1$ the on-shell δ function is reproduced.

One can thus conclude, that only such x^+ orderings with only one state between the two hard vertices have to be considered.

LCPT V: contributing x^+ orderings for DPDs

Consider now a DPD, which can again be decomposed as

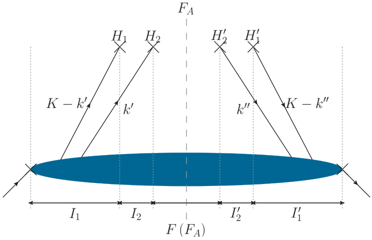
$$\Phi_{DPD} = I_1\, I_2\, F\,(F_A)\; I_2'\, I_1'$$

LCPT V: contributing x^+ orderings for DPDs

Consider now a DPD, which can again be decomposed as

$$\Phi_{DPD} = I_1 \, I_2 \, F\,(F_A)\; I_2' \, I_1'$$

to be able to use the same argument as before consider the following two x^+ orderings



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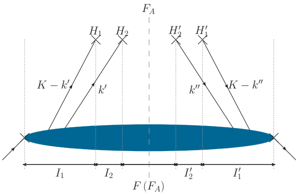

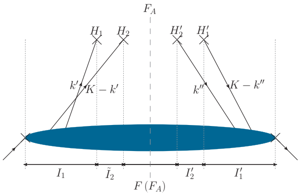

Consider now the states between H_1 and H_2, I_2 and $\tilde{I}_2$

$$I_2 = \frac{1}{p^- - (K^- - k'^-) - D_{I_2} + i\epsilon} \qquad \tilde{I}_2 = \frac{1}{p^- - k'^- - D_{\tilde{I}_2} + i\epsilon}\,.$$

LCPT V: contributing x^+ orderings for DPDs

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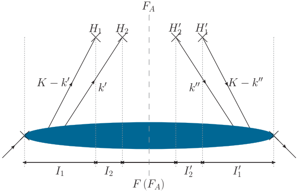
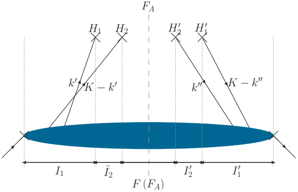
to be able to use the same argument as before consider the following two x^+ orderings





As k'^- only occurs in these energy denominators we can sum these two x^+ orderings and integrate over k'^-

$$\int \frac{\mathrm{d}k^-}{2\pi}\left[I_2+\tilde{I}_2\right]=\int\frac{\mathrm{d}k'^-}{2\pi}\left[\frac{2p^- - K^- - D_{\tilde{I}_2} - D_{I_2}}{\left(p^- - (K^- - k'^-) - D_{I_2}+i\epsilon\right)\left(p^- - k'^- - D_{\tilde{I}_2}+i\epsilon\right)}\right]=-i$$

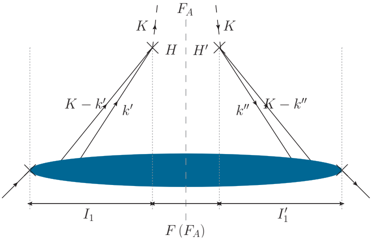
Repeating the same on the rhs yields a factor of i.

LCPT V: contributing x^+ orderings for DPDs

Consider now a DPD, which can again be decomposed as

$$\Phi_{DPD} = I_1\, I_2\, F\, (F_A)\; I_2'\, I_1'$$

Thus we can conclude, that summing over the possible orderings of the hard vertices and integrating over k'^- and k''^- is tantamount to setting the hard vertices on each side of the final state cut to the same x^+ value



LCPT VI: updated PDF and DPD definitions

PDF and DPD definitions

$$f_B^{j_1}(x_1) = \sum_t \sum_c \sum_o \left(x_1\, p^+\right)^{n_1} p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+ \right)$$
$$\times\, \Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right) \delta\Big(1 - \sum_{i=1}^{M(c)} x_i\Big)$$

$$\int_0^{1-x_1} \mathrm{d}x_2\, F_B^{j_1 j_2}(x_1,x_2) = \sum_t \sum_c \sum_o \sum_l \delta_{f(l),j_2} \left(x_1 p^+\right)^{n_1} 2p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+ \right)$$
$$\times\, \left(x_l\, p^+\right)^{n_l} \Phi^{j_1 j_2}_{DPD_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right) \delta\Big(1 - \sum_{i=1}^{M(c)} x_i\Big)$$

LCPT VI: updated PDF and DPD definitions

PDF and DPD definitions

$$f_B^{j_1}(x_1) = \sum_t \sum_c \sum_o \left(x_1\, p^+\right)^{n_1} p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+ \right) \times \Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right) \delta\Big(1 - \sum_{i=1}^{M(c)} x_i\Big)$$

$$\int_0^{1-x_1} \mathrm{d}x_2\, F_B^{j_1 j_2}(x_1, x_2) = \sum_t \sum_c \sum_o \sum_l \delta_{f(l),j_2} \left(x_1 p^+\right)^{n_1} 2p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+ \right) \times \left(x_l\, p^+\right)^{n_l} \Phi^{j_1 j_2}_{DPD_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right) \delta\Big(1 - \sum_{i=1}^{M(c)} x_i\Big)$$

Comparing these expressions, one finds that the rhs is basically the same (neglecting the sum over l) if one can show that

$$2\left(x_l\, p^+\right)^{n_l} \Phi^{j_1,\, j_2}_{DPD_{t,c,o}} = \Phi^{j_1}_{PDF_{t,c,o}}$$

Number Sum Rule

Assuming we have shown that $2\left(x_l\, p^+\right)^{n_l}\, \Phi^{j_1\,,\, j_2}_{DPD_{t,c,o}} = \Phi^{j_1}_{PDF_{t,c,o}}$ the number sum rule can be rewritten as

$$\sum_t \sum_c \sum_o \sum_l \left(\delta_{f(l),\, j_2} - \delta_{f(l),\, \overline{j_2}}\right) \left(x_1\, p^+\right)^{n_1} p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+\right)$$

$$\times\, \Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\}, \{\boldsymbol{k}\}\right) \delta\left(1 - \sum_{i=1}^{M(c)} x_i\right)$$

$$= \left(N_{j_{2v}} + \delta_{j_1\,,\, \overline{j_2}} - \delta_{j_1\,,\, j_2}\right) \sum_t \sum_c \sum_o \left(x_1\, p^+\right)^{n_1} p^+ \int \frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}} \left(\prod_{i=2}^{N(t)} \frac{\mathrm{d}x_i \mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}} p^+\right)$$

$$\times\, \Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\}, \{\boldsymbol{k}\}\right) \delta\left(1 - \sum_{i=1}^{M(c)} x_i\right)$$

Number Sum Rule

Assuming we have shown that $2\left(x_l\, p^+\right)^{n_l}\,\Phi^{j_1\,,\,j_2}_{DPD_{t,c,o}} = \Phi^{j_1}_{PDF_{t,c,o}}$ the number sum rule can be rewritten as

$$\sum_t\sum_c\sum_o\sum_l\left(\delta_{f(l),\,j_2}-\delta_{f(l),\,\overline{j_2}}\right)\left(x_1\,p^+\right)^{n_1}p^+\int\frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}}\left(\prod_{i=2}^{N(t)}\frac{\mathrm{d}x_i\mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}}p^+\right)$$

$$\times\,\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\delta\left(1-\sum_{i=1}^{M(c)}x_i\right)$$

$$=\left(N_{j_{2v}}+\delta_{j_1\,,\,\overline{j_2}}-\delta_{j_1\,,\,j_2}\right)\sum_t\sum_c\sum_o\left(x_1\,p^+\right)^{n_1}p^+\int\frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}}\left(\prod_{i=2}^{N(t)}\frac{\mathrm{d}x_i\mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}}p^+\right)$$

$$\times\,\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\delta\left(1-\sum_{i=1}^{M(c)}x_i\right)$$

which reduces to

$$\sum_l\left(\delta_{f(l),\,j_2}-\delta_{f(l),\,\overline{j_2}}\right)=\left(N_{j_{2v}}+\delta_{j_1\,,\,\overline{j_2}}-\delta_{j_1\,,\,j_2}\right)$$

Momentum Sum Rule

For the momentum sum rule one analogously finds

$$\sum_{j_2}\sum_{t}\sum_{c}\sum_{o}\sum_{l}\delta_{f(l),\,j_2}\left(x_1\,p^+\right)^{n_1}p^+\int\frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}}\left(\prod_{i=2}^{N(t)}\frac{\mathrm{d}x_i\mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}}p^+\right)$$

$$\times\, x_l\,\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\delta\left(1-\sum_{i=1}^{M(c)}x_i\right)$$

$$=(1-x_1)\sum_{t}\sum_{c}\sum_{o}\left(x_1\,p^+\right)^{n_1}p^+\int\frac{\mathrm{d}^{D-2}\boldsymbol{k}_1}{(2\pi)^{D-1}}\left(\prod_{i=2}^{N(t)}\frac{\mathrm{d}x_i\mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}}p^+\right)$$

$$\times\,\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\delta\left(1-\sum_{i=1}^{M(c)}x_i\right)$$

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$$\times\,\Phi^{j_1}_{PDF_{t,c,o}}\left(\{x\},\{\boldsymbol{k}\}\right)\delta\left(1-\sum_{i=1}^{M(c)}x_i\right)$$

using a shorthand notation for the integration measures

$$\int\mathrm{D}^b_a\left[x_i\right]=\prod_{i=a}^{b}\int_0^1\mathrm{d}x_i\,p^+\qquad\qquad\int\mathrm{D}^b_a\left[\boldsymbol{k}_i\right]=\prod_{i=a}^{b}\int\frac{\mathrm{d}^{D-2}\boldsymbol{k}_i}{(2\pi)^{D-1}}\,,$$

Momentum Sum Rule

this can be rewritten as

$$\sum_t \sum_c \sum_o \sum_l \left(x_1\, p^+\right)^{n_1} p^+ \int \mathrm{D}_2^{N(t)}[x_i]\, \mathrm{D}_1^{N(t)}[\boldsymbol{k}_i]\, x_l\, \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\left(1 - \sum_{i=1}^{M(c)} x_i\right)$$

$$= (1 - x_1) \sum_t \sum_c \sum_o \left(x_1\, p^+\right)^{n_1} p^+ \int \mathrm{D}_2^{N(t)}[x_i]\, \mathrm{D}_1^{N(t)}[\boldsymbol{k}_i]\, \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\left(1 - \sum_{i=1}^{M(c)} x_i\right)$$

Momentum Sum Rule

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$$= (1-x_1) \sum_t \sum_c \sum_o (x_1\, p^+)^{n_1} p^+ \int \mathrm{D}_2^{N(t)}[x_i]\, \mathrm{D}_1^{N(t)}[\boldsymbol{k}_i]\, \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\left(1-\sum_{i=1}^{M(c)} x_i\right)$$

which reduces to

$$\sum_l \int \mathrm{D}_2^{N(t)}[x_i]\, \mathrm{D}_1^{N(t)}[\boldsymbol{k}_i]\, x_l\, \Phi^{j_1}_{PDF_{t,c,o}}(\{x\},\{\boldsymbol{k}\})\, \delta\left(1-\sum_{i=1}^{M(c)} x_i\right)$$

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