SSB in Hyperbolic Field Theory

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R. Cartas-Fuentevilla and O. Meza Aldama, Eur.Phys.J. C **76** (2016) no.2, 98, arXiv:1506.04410 [hep-th].

May 25th, 2016.

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Hyperbolic numbers

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Define

$$\mathbb{D} \equiv \{x + jy | x, y \in \mathbb{R}\} \tag{1}$$

Properties

For hyperbolic numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$,

$$z_1 \pm z_2 = (x_1 + jy_1) \pm (x_2 + jy_2) = (x_1 \pm x_2) + j(y_1 \pm y_2), \quad (2)$$

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Versors:

$$e^{j\alpha} \equiv \cosh\alpha + j\sinh\alpha. \quad (7)$$

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The lagrangian

Consider

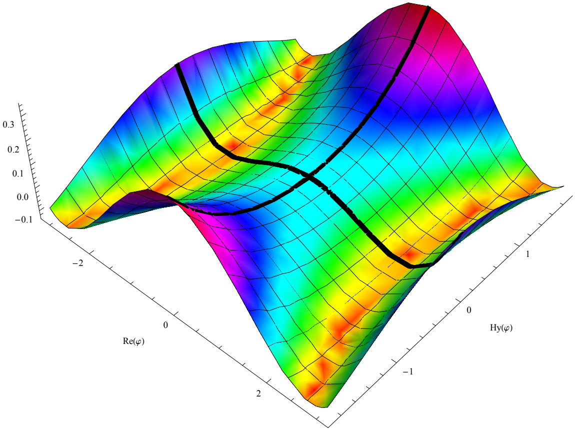
$$\mathscr{L} = \partial^\mu \bar{\varphi} \partial_\mu \varphi - V(\varphi, \bar{\varphi}), \tag{8}$$

with

$$V(\varphi, \bar{\varphi}) = m^2 \varphi \bar{\varphi} + \frac{\lambda}{2} (\varphi \bar{\varphi})^2, \tag{9}$$

where $\varphi(x) \in \mathbb{D}$.

The potential



Negative squared mass.

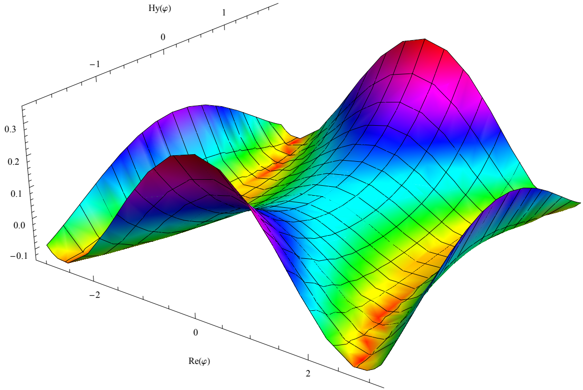


Figure: Potential with $m^2 < 0$

Positive squared mass.

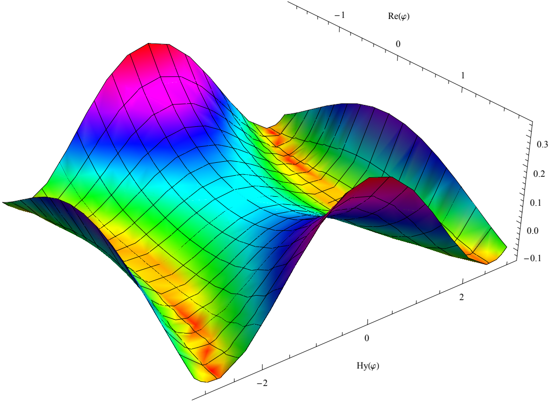


Figure: Potential with $m^2 > 0$

Spontaneous symmetry breaking.

Write $\varphi = \varphi_1 + j\varphi_2$

$$\Rightarrow V(\varphi_1, \varphi_2) = m^2(\varphi_1^2 - \varphi_2^2) + \frac{\lambda}{2}(\varphi_1^2 - \varphi_2^2)^2. \qquad (10)$$

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In terms of which

$$V(\chi_1, \chi_2) = 2m^2\chi_1^2 + \frac{\lambda}{2}\chi_1^4 + \frac{\lambda}{2}\chi_2^4 \tag{13}$$

$$-\lambda\chi_1^2\chi_2^2 + 2\lambda K\chi_1(\chi_1^2 - \chi_2^2). \tag{14}$$

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Bicomplex numbers

Now define $z \in \mathbb{H}$ such that:

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Bicomplex phases:

$$\begin{aligned} e^{i\alpha + j\beta} &\equiv e^{i\alpha} e^{j\beta} \\ &= \cos\alpha \cosh\beta + i \sin\alpha \cosh\beta + j \cos\alpha \sinh\beta + ij \sin\alpha \sinh\beta. \end{aligned} \tag{18}$$

Restriction

Impose:

$$x = \gamma w \quad , \quad y = \gamma v, \tag{19}$$

with $\gamma \in \mathbb{R}$. (It is the only consistent way.)

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Then

$$z = (\gamma + ij)w + (i\gamma + j)v \quad , \quad \bar{z} = (\gamma + ij)w - (i\gamma + j)v, \tag{20}$$

$$|z|^2 = (\gamma^2 - 1)(v^2 + w^2) + 2ij\gamma(w^2 - v^2). \tag{21}$$

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Bicomplex lagrangian

Consider

$$\mathscr{L} = \partial^\mu \bar{\psi} \partial_\mu \psi - V(\psi, \bar{\psi}), \tag{22}$$

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where $\psi \in \mathbb{H}$, $a = \pm 1$ and

$$m^2 = m_R^2 + ijm_H^2, \tag{24}$$

$$\lambda = \lambda_R + ij\lambda_H, \tag{25}$$

so the potential is "bounded".

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$$V_R = a\left(\frac{\gamma^2-1}{2}m_R^2 + \gamma m_H^2\right)v^2 + a\left(\frac{\gamma^2-1}{2}m_R^2 - \gamma m_H^2\right)w^2 + \frac{\lambda_R}{6}\left[\frac{(\gamma^2-1)^2}{4}(v^2+w^2)^2 - \gamma^2(v^2-w^2)^2\right] - \frac{\lambda_H}{6}\gamma(\gamma^2-1)(w^4-v^4), \tag{26}$$

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$$V_H = a\left(\frac{\gamma^2-1}{2}m_H^2 - \gamma m_R^2\right)v^2 + a\left(\frac{\gamma^2-1}{2}m_H^2 + \gamma m_R^2\right)w^2 + \frac{\lambda_R}{6}\gamma(\gamma^2-1)(w^4-v^4) + \frac{\lambda_H}{6}\left[\frac{(\gamma^2-1)^2}{4}(v^2+w^2)^2 - \gamma^2(v^2-w^2)^2\right], \quad (27)$$

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which is equivalent to

$$(1-\gamma^2)(\lambda_R^2+\lambda_H^2)(v_0^2+w_0^2) = 6a(\lambda_R m_R^2 + \lambda_H m_H^2), \tag{30}$$

$$\gamma(\lambda_R^2+\lambda_H^2)(v_0^2-w_0^2) = 3a(\lambda_R m_H^2 - \lambda_H m_R^2). \tag{31}$$

Minimization (2)

This is a minimum if (for example)

$$v_0^2 = \frac{6am_R^2}{(1-\gamma^2)\lambda_R + 2\gamma\lambda_H}, \quad (32)$$

$$w_0 = 0. \quad (33)$$

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$$\Rightarrow \frac{m_H^2}{m_R^2} = \frac{(1-\gamma^2)\lambda_H + 2\gamma\lambda_R}{(1-\gamma^2)\lambda_R - 2\gamma\lambda_H}. \tag{34}$$

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There are several cases to study.

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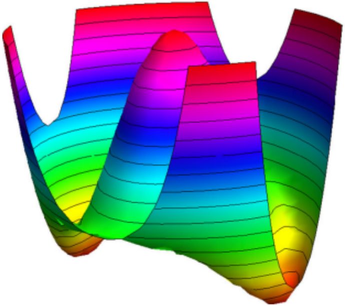
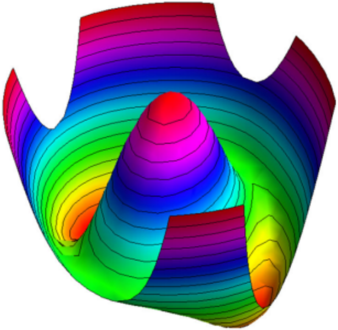
Constraints on γ:

$$\gamma \in \sim (-0.2, 0.2). \tag{37}$$

Potential in the allowed region





Mass generation

The quadratic terms in the potential reduce to

$$a\left(\frac{1-\gamma^2}{2}m_R^2-\gamma m_H^2\right)v^2+a\left(\frac{1-\gamma^2}{2}m_R^2+\gamma m_H^2\right)w^2 \quad (38)$$

$$+\,ija\left[a\left(\frac{1-\gamma^2}{2}m_H^2+\gamma m_R^2\right)v^2+a\left(\frac{1-\gamma^2}{2}m_H^2-\gamma m_H^2\right)w^2\right] \quad (39)$$

$$=-\,ija\frac{m_R^2}{2}P_H^v v^2, \quad (40)$$

where

$$P_H^v=\frac{\gamma^4-4\gamma^3-6\gamma^2+4\gamma+1}{\gamma^2+2\gamma-1}. \quad (41)$$

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- Bicomplex field leads to a deformation of the mexican hat potential