A worldline approach to QCD amplitudes

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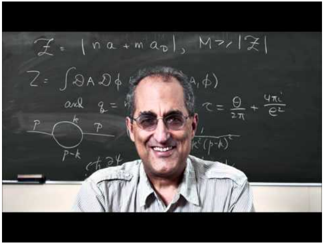
26 August

- Edward Witten's birthday



As for the forces, electromagnetism and gravity we experience in everyday life. But the weak and strong forces are beyond our ordinary experience. So in physics, lots of the basic building blocks take 20th- or perhaps 21st-century equipment to explore. –E. Witten

- EW has been a pioneer of the worldline formalism: used particle models to compute (gravitational) anomalies Alvarez-Gaume, Witten 1985

Outline

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1. Introduction
2. QED in the Worldline Formalism
   - Tree-level
   - One-loop
3. QCD in the Worldline Formalism
   - Colored particle
   - One-loop
   - Tree-level

Outline

Worldline methods: Quantum Field Theory results from qzn of QM models

- Main tools in use: particle actions

$$\text{(schematically)} \quad S[x,\psi;G] = \int_0^T d\tau \Big(\dot{x}^2 + \psi\dot{\psi} + V(x,\dot{x},\psi;G) \Big)$$

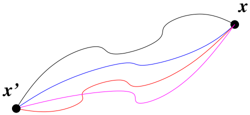
x bosonic $\quad \psi$ fermionic $\quad G$ external

- canonical qzn
- path integral (integral over trajectories)

Purely bosonic

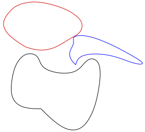
- Dirichlet boundary conditions (topology of a line)

$$\langle x|e^{-TH}|x'\rangle = \int_{x(0)=x'}^{x(T)=x} Dx(\tau)e^{-S[x;G]}$$



- Periodic boundary conditions (topology of a circle)

$$Z(T) = \int_{x(0)=x(T)} Dx(\tau)e^{-S[x;G]}$$



Quantum Field Theory

- Second quantization: computation of Feynman diagrams from correlation functions of *fields*
- In perturbation theory the most generic diagram can be built from propagators (Green functions) and one particle irreducible diagrams (effective vertices)

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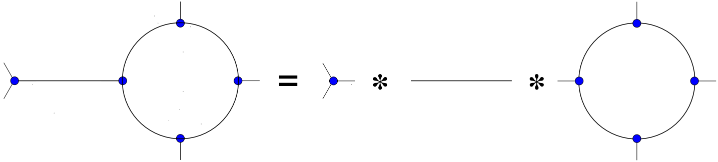
- Example: scalar theory with cubic interaction $\lambda\phi(x)^3$



- Effective vertices are the key objects for renormalization

Worldline Formalism

Tool to compute

- Green functions (propagators)
- effective actions, i.e. functional generators of effective vertices

using particle models

review by Schubert '01

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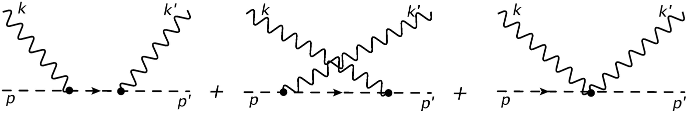
Why do we want to study other tools?

- in some cases QFT is not the most effective way to compute
- other methods have proved very successful in the computation of S-matrix elements:
  - MHV amplitudes, holomorphic methods,... Bern, Kosower, Cachazo,.... mostly at tree level with massless particle

  worldline formalism works well *also* with massive particle and at one loop

Advantages

- no need to compute momentum integrals or Dirac traces explicitly
- efficient way to path order
- directly obtain off-shell Feynman amplitudes, rather than single Feynman diagrams.
  Ex: Compton scattering in scalar QED



- gauge-invariance efficiently guaranteed
- allows presence of (strong) external fields

WF: Green's function

$$\big(-\partial_\mu \partial^\mu + m^2 \big) \Delta(x, x') = \delta(x - x') \quad \equiv$$

- Massive scalar field (Feynman) propagator

$$\Delta(x, x') = \langle \phi(x) \phi(x') \rangle = \int d^4p \, \frac{e^{-ip\cdot(x-x')}}{p^2 + m^2}$$

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- Schwinger representation

$$\begin{aligned}\langle \phi(x)\phi(x')\rangle &= \int_0^\infty dT \int d^4p \; e^{-ip\cdot(x-x')-T(p^2+m^2)} \\ &= \int_0^\infty dT \int d^4p \; \langle x| e^{-T(p^2+m^2)}|p\rangle\langle p|x'\rangle\end{aligned}$$

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- Replacing p with $\mathbb{p}$ can integrate over p

WF: Green's function

$$\langle\phi(x)\phi(x')\rangle = \int_0^\infty dT\ e^{-Tm^2}\langle x|e^{-T\mathbb{p}^2}|x'\rangle\,, \quad \mathbb{H} = \mathbb{p}^2 = \delta_{\mu\nu}\mathbb{p}^\mu\mathbb{p}^\nu$$

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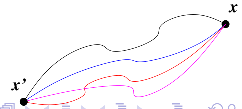
- Path integral representation of particle transition amplitude

$$\langle\phi(x)\phi(x')\rangle = \int_0^\infty dT\ e^{-Tm^2}\int_{x(0)=x'}^{x(T)=x} Dx\ e^{-S[x]} \tag{1}$$

$$S[x(\tau)] = \frac{1}{4T}\int_0^1 d\tau\ \delta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \tag{2}$$

(1) Worldline representation for the free scalar field propagator = path integral on the line
(2) Worldline action



Tree-level scalar QED

Coupling to external photons: replace $\mathbb{p}_\mu$ with $\Pi_\mu = \mathbb{p}_\mu - qA_\mu$

$$\langle \phi(x)\bar{\phi}(x')\rangle_A = \int_0^\infty dT\ e^{-Tm^2} \int_{x(0)=x'}^{x(T)=x} Dx\ e^{-S[x,A_\mu]}$$

$$S[x(\tau), A_\mu] = \int_0^1 d\tau\ \left(\frac{1}{4T}\delta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + iq\dot{x}^\mu A_\mu(x(\tau))\right)$$

gauge invariant

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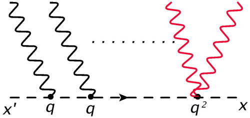
gauge invariant

- treat A_μ perturbatively: the latter reproduces all tree-level diagrams of scalar QED with 2 scalars and N photons, the "N-propagator"

$$\langle \phi(x)\bar{\phi}(x')\rangle_A = \sum$$



- the WL linear coupling also reproduces the sea-gull coupling of QFT

Tree-level scalar QED

Recipe:

- Write potential as trivial background plus sum of photons

$$A_\mu(x(\tau)) = \sum_{i=1}^{n} \epsilon_{i,\mu} e^{ik_i \cdot x(\tau)}$$

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- expand $e^{-iq\int \dot{x}\cdot A}$ and pick up terms linear in all polarizations: it involves a QM correlation function

$$\mathcal{A}[x, x', k_1, \epsilon_1; \cdots ; k_n, \epsilon_n] = q^n \int_0^\infty dT e^{-Tm^2} \prod_{i=1}^{n} \int_0^1 d\tau_i$$

$$\times \int_{x(0)=x'}^{x(1)=x} Dx \; e^{-\frac{1}{4T}\int \dot{x}^2} e^{\sum_i \epsilon_i \cdot \dot{x}(\tau_i) + ik_i \cdot x(\tau_i)} \Big|_{m.l.\epsilon_i}$$

Tree-level scalar QED

Recipe (cont'd):

- split $x(\tau) = x_{bg}(\tau) + y(\tau)$ con $y(0) = y(1) = 0$

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- split $x(\tau) = x_{bg}(\tau) + y(\tau)$ con $y(0) = y(1) = 0$
- then $\langle y^\mu(\tau) y^{\mu'}(\tau')\rangle = \mathcal{N}\int_{y(0)=0}^{y(1)=0} Dx\; e^{-\frac{1}{4T}\int \dot{y}^2} = -2T\delta^{\mu\mu'}\Delta(\tau,\tau')$

  $\Delta(\tau,\tau')$ particle Green's function

$$\mathcal{A}[x, x', k_1, \epsilon_1; \cdots; k_n, \epsilon_n] = q^n \int_0^\infty \frac{dT}{(4\pi T)^2} e^{-Tm^2 - \frac{1}{4T}(x-x')^2} \prod_{i=1}^n \int_0^1 d\tau_i$$

$$e^{\sum_i \left(ik_i\cdot x' + (i\tau_i k_i + \epsilon_i)\cdot(x-x')\right)} e^{T\sum_{i,j}\left(k_i\cdot k_j \Delta_{ij} - i2\epsilon_i\cdot k_j {}^\bullet\Delta_{ij} - \epsilon_i\cdot\epsilon_j {}^\bullet\Delta^\bullet_{ij}\right)}\Big|_{m.l.\epsilon_i}$$

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- to get a full momentum amplitude, use Fourier transform $\int dx dx' e^{i(p\cdot x + p'\cdot x')}$

Tree-level scalar QED - full momentum amplitude

Recipe (cont'd):

- the integral over the "center of mass" $x_+ = \frac{x+x'}{2}$ gives the energy-momentum delta function $\delta(p + p' + \sum k)$, the integral over the "distance" is Gaussian

$$\tilde{\mathcal{A}}[p, p', k_1, \epsilon_1; \cdots; k_n, \epsilon_n] = q^n \int_0^\infty dT\ e^{-T(m^2+p^2)} \prod_{i=1}^{n} \int_0^1 d\tau_i$$

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where $\Delta_{ij} = \frac{1}{2}|\tau_i - \tau_j|, \quad \Rightarrow \quad \dot{\Delta}_{ij} = \frac{1}{2}\mathrm{sign}(\tau_i - \tau_j), \quad \ddot{\Delta}_{ij} = \delta(\tau_i - \tau_j)$

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- integrals over T and τ_i are the Feynman parametrization of scalar free propagators
- on-shell the integrand is fully τ-translation invariant
- the external scalar lines aren't (yet) truncated

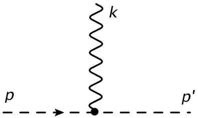
Tree-level scalar QED - Examples

- Simplest case: n=1

$$\mathcal{A}(p,p';k,\epsilon) = q\delta(p+p'+k)\epsilon\cdot(p-p')$$
$$\int_0^\infty dT\ e^{-T(m^2+p^2)}\int_0^T dt\ e^{-t(p-p')\cdot k}$$
$$= q\delta(p+p'+k)\frac{\epsilon\cdot(p-p')}{(p^2+m^2)(p'^2+m^2)}$$

- Upon truncation

$$\mathcal{M}(p,p';k,\epsilon) = \delta(p+p'+k)q\ \epsilon\cdot(p-p') =$$



- On shell it vanishes

Tree-level scalar QED - Examples

- Compton Scattering (n=2)

$$\mathcal{A}(p,p';k,\epsilon,k;\epsilon',k') = q^2\delta(p+p'+k+k')\int_0^\infty dT\; e^{-T(m^2+p^2)}$$

$$\int_0^T dt\int_0^T dt'\; e^{-(p-p')\cdot(kt+k't')+k\cdot k'|t-t'|}\Big[\epsilon\cdot\epsilon'\delta(t-t')$$

$$+\epsilon\cdot((p-p')-k'\mathrm{sgn}(t-t'))\epsilon'\cdot((p-p')+k\mathrm{sgn}(t-t'))\Big]$$

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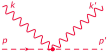
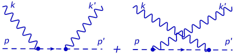
$$+\epsilon\cdot((p-p')-k'\mathrm{sgn}(t-t'))\epsilon'\cdot((p-p')+k\mathrm{sgn}(t-t'))\Big]$$

- Term with $\delta(t-t')$ reproduces the seagull diagram 

- Other terms reproduce the two two-vertex diagrams, one when $t>t'$, the other when $t<t'$ 

- the full amplitude is gauge invariant

Scalar QED one-loop effective action

- Effective action: functional generator of 1PI correlation functions
- Example: scalar QED

$$S[\phi; A] = \int d^D x \left[|D_\mu(A)\phi|^2 + m^2 |\phi|^2 \right]$$

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- One-loop effective action (singles out the quadratic part)

$$e^{-\Gamma[A]} = \int D\phi D\phi^* \; e^{-S[\phi;A]} = \mathrm{Det}^{-1}\left[-D^2(A) + m^2 \right]$$

$$\Gamma[A] = \mathrm{Tr}\ln\left[-D^2(A) + m^2 \right], \quad D_\mu(A) = \partial_\mu + iqA_\mu$$

$$= \int_0^\infty \frac{dT}{T} \mathrm{Tr}\; e^{-T[-D^2(A)+m^2]}$$

$$= \int_0^\infty \frac{dT}{T} \int dx \langle x | e^{-T[-D^2(A)+m^2]} | x \rangle$$

Scalar QED one-loop effective action

- Quantum hamiltonian $H = -\frac{1}{2}D^2(A) = \frac{1}{2}\Pi^2(A)$
- Worldline representation of one-loop effective action

$$\Gamma[A] = \int_0^\infty \frac{dT}{T} e^{-Tm^2} \oint Dx\ e^{-S[x,A]}$$

$$S[x,A] = \int_0^1 d\tau \left[\frac{1}{4T}\delta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + iq\dot{x}^\mu A_\mu(x(\tau))\right]$$

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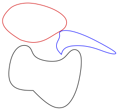
- sum over all closed trajectories $x(0) = x(1)$: topology of circle



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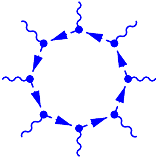
- $\Gamma[A]$ yields photon amplitudes

$$\Gamma[A] = \sum$$



Photon amplitudes from effective action

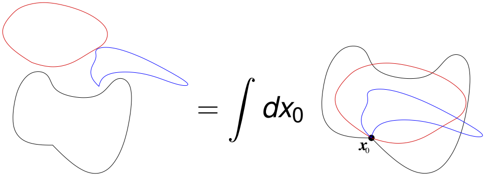
In momentum space

- Write potential as a sum of photons and pick up terms linear in all polarizations

$$\Gamma[k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty \frac{dT}{T} e^{-Tm^2} \prod_{i=1}^n \int_0^1 d\tau_i$$
$$\times \oint Dx\; e^{-\frac{1}{4T}\int \dot{x}^2} e^{\sum_i \epsilon_i\cdot\dot{x}(\tau_i)+ik_i\cdot x(\tau_i)}\Big|_{m.l.\epsilon_i}$$

- Factor out a zero mode $x(\tau) = x_0 + y(\tau)$

$$\oint Dx\; e^{-\frac{1}{4T}\int \dot{x}^2} = \int dx_0 \int Dy\; e^{-\frac{1}{4T}\int \dot{y}^2}$$



Photon amplitudes from effective action

Bern-Kosower master formula

$$\Gamma[k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty \frac{dT}{T}\frac{e^{-Tm^2}}{(4\pi T)^{D/2}}\prod_{i=1}^n \int_0^1 d\tau_i \int dx_0\, e^{ix_0\cdot\sum p_i}$$

$$e^{T\sum_{i,j}\left(k_i\cdot k_j\Delta_{ij} - i2\epsilon_i\cdot k_j {}^\bullet\Delta_{ij} - \epsilon_i\cdot\epsilon_j {}^\bullet\Delta^\bullet_{ij}\right)}\Big|_{m.l.\epsilon_i}$$

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- path integral normalization $\frac{1}{(4\pi T)^{D/2}} = \int Dy\ e^{-\frac{1}{4T}\int \dot{y}^2}$
- momentum conservation $\int dx_0 e^{ix_0\cdot\sum p_i} = \delta(\sum p_i)$
- $\int dT \prod_i \int d\tau_i$ give the Feynman-integral representation of the loop diagram (including the momentum integral)

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Bern-Kosower master formula

$$\Gamma[k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty \frac{dT}{T}\frac{e^{-Tm^2}}{(4\pi T)^{D/2}}\prod_{i=1}^n \int_0^1 d\tau_i \int dx_0 e^{ix_0\cdot\sum p_i}$$

$$e^{T\sum_{i,j}\left(k_i\cdot k_j\Delta_{ij} - i2\epsilon_i\cdot k_j {}^\bullet\Delta_{ij} - \epsilon_i\cdot\epsilon_j {}^\bullet\Delta^\bullet_{ij}\right)}\Big|_{m.l.\epsilon_i}$$

- path integral normalization $\frac{1}{(4\pi T)^{D/2}} = \int Dy\; e^{-\frac{1}{4T}\int \dot{y}^2}$
- momentum conservation $\int dx_0 e^{ix_0\cdot\sum p_i} = \delta(\sum p_i)$
- $\int dT \prod_i \int d\tau_i$ give the Feynman-integral representation of the loop diagram (including the momentum integral)
- Bern and Kosower '91 derived it from $\alpha' \to 0$ of string amplitudes
- Strassler '92 rederived BK formula directly from 1-st quantized QFT (as done above)

Spinor QED one-loop effective action

- A (charged) Dirac field coupled to electromagnetism

$$S_D = \int \bar{\psi}\Big(\not{D}(A) + m\Big)\psi$$

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$$= \frac{1}{2}\mathrm{Tr}\log\Big(-D(A)^2 + q\gamma^{\mu\nu}F_{\mu\nu} + m^2\Big)$$

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- " tr" is over fermionic indices, $\mathcal{P}$ is the path-ordering (action is matrix-valued): perturbative expansion is trickier

Summary

- Abelian particle path integral on the line $\longrightarrow$ Scalar QED propagator and tree-level amplitudes
- Abelian path integral on the circle $\longrightarrow$ Scalar QED effective action and one-loop amplitudes
- Abelian path integral on the circle with spin factor $\longrightarrow$ Spinor QED effective action and one-loop amplitudes

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- Abelian particle path integral on the line $\longrightarrow$ Scalar QED propagator and tree-level amplitudes
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- Spinorial DoF's can be generated dynamically $\longrightarrow$ perturbative expansion for WL Spinor QED effective action simplified and no need for explicit path-ordering

Abelian Spinning particle

- Take Grassmann coordinates, ψ^μ with $S_f = \frac{i}{2}\int d\tau \psi_\mu \dot{\psi}^\mu$
- Canonical quantization $\{\hat{\psi}_\mu, \hat{\psi}_\nu\} = \delta_{\mu\nu} \;\Rightarrow\; \hat{\psi}_\mu \sim \gamma_\mu$

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- The locally susy action:
  $S[x,p,\psi,e,\chi;A] = \int d\tau \Big[p\cdot\dot{x} + \frac{i}{2}\psi\cdot\dot{\psi} - eH - i\chi Q \Big]$
  with $Q = \psi\cdot\Pi(A)$
  and $H = \frac{1}{2}\Pi^2 = \frac{i}{2}\{Q,Q\}_{db}$
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- In Euclidean configuration space $S[x, \psi, e, \chi; A] =$
  $\int d\tau \Big[\frac{1}{2e}(\dot{x}^\mu - \chi\psi^\mu)^2 + \frac{1}{2}\psi_\mu \dot{\psi}^\mu + iq\dot{x}^\mu A_\mu + \frac{e}{2} qF_{\mu\nu}\psi^\mu\psi^\nu\Big]$
  gauge-invariant
- The old spin factor $\gamma^{\mu\nu}F_{\mu\nu}$ turned into $\psi^\mu\psi^\nu F_{\mu\nu}$
- The ψ_μ correlators reproduce the $\mathrm{tr}\,\mathcal{P}$

Spinor QED effective action

On the circle $e = 2T,\ \chi = 0$ and $A_\mu(x) = \sum_i \epsilon_{i,\mu} e^{ik_i \cdot x}$

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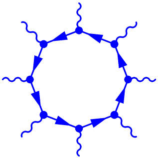
- photon amplitudes are computed as spinning worldline correlation functions (x's have PBC, ψ's have ABC)

$$\Gamma[k_1,\epsilon_1;\cdots;k_n,\epsilon_n] = q^n \int_0^\infty \frac{dT}{T} \prod_{i=1}^n \int_0^1 d\tau_i$$

$$\times \oint Dx\,D\psi\ e^{-\int\left(\frac{1}{4T}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi}\right)} e^{\sum_i \epsilon_i\cdot\dot{x}(\tau_i)+ik_i\cdot x(\tau_i)+T\epsilon\cdot\psi p\cdot\psi(\tau_i)}\Big|_{m.l.\epsilon_i}$$

$=$ 

- massive theory: KK reduction from a D+1 massless theory

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$\mathcal{P}e^{-\int_0^1 d\tau\left(\frac{1}{4T}\dot{x}^2 + ig\dot{x}\cdot W\right)}$ gauge-covariant on the line

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  $\mathcal{P}$ path-ordering, Tr_c trace over color indices
- Similarly to the fermionic DoF's, auxiliary fields can be used

Formalism

- non-abelian coupling $S_{int}[x; W] = ig \int d\tau \dot{x}^\mu W_\mu^a \big(T_a\big)_\alpha{}^{\alpha'}$ with $\alpha = 1, \ldots, N$ of $G = SU(N)$

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- kinetic term $S[c] = \int d\tau \bar{c}^\alpha \dot{c}_\alpha$
- canonical qzn $[c_\alpha, \bar{c}^{\alpha'}] = \delta_\alpha^{\alpha'}$ harmonic oscillators
- Fock space $|0\rangle$, $\bar{c}^\alpha|0\rangle, \bar{c}^{\alpha_1}\bar{c}^{\alpha_2}|0\rangle$,...
- representation is reducible: $\bullet + \square + \square\square + \cdots$

Formalism

- the particle action is invariant under a U(1) symmetry
  $c(\tau) \to e^{i\alpha} c(\tau)$, $\bar{c}(\tau) \to \bar{c}(\tau) e^{-i\alpha}$
- can restrict to a fixed occupation number; e.g. $r = 1$, i.e. $\square$
  (i) add a worldline gauge field $a(\tau)$, i.e. make the U(1) symmetry local Bastianelli, Bonezzi, OC, Latini 2013, 2015
  (ii) add a Chern Simons term

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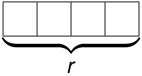
- choosing $s = r + \frac{N}{2}$ produces occupation number r, i.e. 

Formalism

- On a generic wave function (in a coherent state basis)
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  $0 = \big(\mathbb{C} - s\big)\Phi \;\Rightarrow\; \Phi = \frac{1}{r!}\phi_{\alpha_1\ldots\alpha_r}(x)\bar{c}^{\alpha_1}\cdots\bar{c}^{\alpha_r}$

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$$\int_0^{2\pi}\frac{d\theta}{2\pi}e^{is\theta}\int_{C,L} Dx \int_{TBC} D\bar{c}Dc\; e^{-S_t[x,c,2T,\theta;W]}$$

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- U(1) aux gauge field a can be gauge-fixed to a constant angle θ, and e to $2T$

Scalar QCD effective action

$$\Gamma[W] = \int_0^\infty \frac{dT}{T} e^{-Tm^2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{is\theta} \int_{PBC} Dx \int_{TBC} D\bar{c}Dc\; e^{-S_t[x,c,2T,\theta;W]}$$

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- coupling of θ to $c(\bar{c})$ absorbed into the boundary conditions

$$\int_{PBC} D\bar{c}Dc\, e^{-\int \bar{c}(\partial_\tau + i\theta)c} = \int_{TBC} D\bar{c}Dc\, e^{-\int \bar{c}\dot{c}} = \left(2i \sin\frac{\theta}{2}\right)^{-N}$$

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- *TBC* means $c(1) = e^{i\theta}c(0),\ \bar{c}(1) = e^{-i\theta}c(0)$

$$\langle c_\alpha(\tau)\bar{c}^\beta(\sigma)\rangle = \delta_\alpha^\beta \Delta(\tau - \sigma, \theta)$$

and

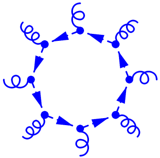
$$S_t[x, c, 2T, \theta; W] = \int d\tau \left[\frac{1}{4T}\dot{x}^2 + ig\dot{x}^\mu A_\mu^a(x)\bar{c}\cdot(T^a)\cdot c + \bar{c}^\alpha \dot{c}_\alpha\right]$$

One-loop gluon amplitudes from effective action

$$\Gamma[k_1,\epsilon_1,a_1,...,k_n,\epsilon_n,a_n] = g^n\delta\left(\sum k\right)\int_0^\infty \frac{dT}{T}\frac{e^{-Tm^2}}{(4\pi T)^{D/2}}\prod_{i=1}^{N}\int_0^1 d\tau_i$$

$$\times\, e^{T\sum_{i,j}\left(k_i\cdot k_j\Delta_{ij} - i2\epsilon_i\cdot k_j{}^{\bullet}\Delta_{ij} - \epsilon_i\cdot\epsilon_j{}^{\bullet}\Delta^{\bullet}_{ij}\right)}\Big|_{m.l.\epsilon_i}$$

$$\times \int_0^{2\pi}\frac{d\theta}{2\pi}\frac{e^{is\theta}}{(2i\sin\frac{\theta}{2})^N}\left\langle\prod_l^n \bar{c}(\tau_l)\cdot T^{a_l}\cdot c(\tau_l)\right\rangle$$

$=$ 

- apart from the green color factor it is the same as the photon amplitude
- extension to spinor QCD straightforward with spinning particle

Scalar QCD propagator

- aux fields have as well Dirichlet b.c.'s: they carry the color states of the in and out scalars Ahmadiniaz, Bastianelli, OC, few days ago

$$\langle \phi(x,\bar{u})\bar{\phi}(x',u')\rangle_W = \int_0^\infty dT\, e^{-Tm^2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{is\theta} \int_{x(0)=x'}^{x(1)=x} Dx \int_{c(0)=u'}^{\bar{c}(1)=e^{-i\theta}\bar{u}} D\bar{c}Dc\, e^{-S_t[x,c,2T,\theta;W]+\bar{c}c(1)}$$

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- split $c(\tau)=u'+\kappa(\tau)$, $\bar c(\tau)=e^{-i\theta}\bar u+\bar\kappa(\tau)$, with $\kappa(0)=\bar\kappa(1)=0$

$$\int_{c(0)=u'}^{\bar c(1)=e^{-i\theta}\bar u} D\bar c Dc\; e^{-\int \bar c\dot c+\bar c c(1)}\;\ldots$$
$$= e^{e^{-i\theta}\bar u\cdot u'}\underbrace{\int_{\kappa(0)=0}^{\bar\kappa(1)=0} D\bar\kappa D\kappa\; e^{-\int\bar\kappa\dot\kappa}}_{=1}\Big\langle\cdots\Big\rangle$$

with $\langle\kappa_\alpha(\tau)\bar\kappa^\beta(\sigma)\rangle=\delta_\alpha^\beta\theta(\tau-\sigma)$

Tree-level amplitudes from Scalar QCD propagator

Recipe is the same as before: $W = \sum \mathrm{gluons}$, multilinearity in ϵ's, Fourier transform the external scalar lines

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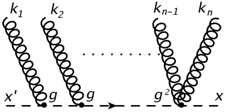
$$\mathcal{A}[p, u, p' u', k_1, \epsilon_1, a_1, ..., k_n, \epsilon_n, a_n]$$

$$= g^n \delta(p + p' + \sum k) \int_0^\infty dT\, e^{-Tm^2} \prod_{l=1}^n \int_0^1 d\tau_l$$

$$\times\, e^{T(p-p')\cdot\sum_l(-k_l\tau_l + i\epsilon_l)} e^{T\sum_{l,l'}(k_l\cdot k_{l'}\Delta_{ll'} - i2\epsilon_l\cdot k_{l'}{}^\bullet\Delta_{ll'} - \epsilon_l\cdot\epsilon_{l'}{}^\bullet\Delta^\bullet_{ll'})}\Big|_{m.l.\epsilon_l}$$

$$\times \int_0^{2\pi} \frac{d\theta}{2\pi} e^{ir\theta + e^{-i\theta}\bar{u}u'} \Big\langle \prod_{l=1}^n \big(e^{-i\theta}\bar{u} + \bar{\kappa}(\tau_l)\big) \cdot T^{a_l} \cdot \big(u + \kappa(\tau_l)\big) \Big\rangle$$

$$= \sum$$ 

- The correlator of auxiliary fields gives the color factor

Ward identities

- The worldline action is gauge invariant provided

$$\widetilde{W}_\mu(x) = U(x)\left[\frac{i}{g}\partial_\mu + W_\mu(x)\right]U^\dagger(x)$$

$$\widetilde{c}(\tau) = U(x(\tau))c(\tau), \quad \widetilde{\bar{c}}(\tau) = \bar{c}(\tau)U^\dagger(x(\tau))$$

Ward identities

- The worldline action is gauge invariant provided

$$\widetilde{W}_\mu(x) = U(x)\left[\frac{i}{g}\partial_\mu + W_\mu(x)\right]U^\dagger(x)$$

$$\widetilde{c}(\tau) = U(x(\tau))c(\tau), \quad \widetilde{\bar{c}}(\tau) = \bar{c}(\tau)U^\dagger(x(\tau))$$

that implies the Ward identity generator (for $r = 1$)

$$0 = D_\mu(W)\frac{\delta}{\delta W_\mu^a(y)}\Big\langle \phi_\alpha(x)\bar{\phi}^\beta(x')\Big\rangle_W$$
$$+ig\delta(y-x)(T^a)_\alpha^{\tilde{\alpha}}\Big\langle \phi_{\tilde{\alpha}}(x)\bar{\phi}^\beta(x')\Big\rangle_W$$
$$-ig\delta(y-x')\Big\langle \phi_\alpha(x)\bar{\phi}^{\tilde{\beta}}(x')\Big\rangle_W (T^a)_{\tilde{\beta}}^\beta$$

- to lowest order

$$\mathcal{A}_{2s,1g}(p,\alpha;p',\beta;-ik,k,a) + ig(T^a)_\alpha{}^\beta(p^2 - p'^2) = 0$$

verified

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  - Compact formula for n-gluon amplitudes
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Outlook:

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Outlook:

- Fields with spin: (non)abelian spinning particles
- Bound states. Done for scalar fields Bastianelli, Huet et al 2014
- Curved space, i.e. perturbative quantum gravity
- KLT relations between graviton amplitudes and gauge amplitudes