Quantum information approach to the description of quantum phase transitions

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Guy Paic and the ICN

- 1996-1998, new development plan of the ICN.
- Creation of the Department:
  High energy physics.
- 2001-2002, Guy agreed to come to Mexico at the ICN.
- Cátedra Patrimonial de Excelencia Nivel II (CONACyT).
- Purpose: Create a laboratory to support measurements and test of detectors mainly related with the ALICE experiment.
- April 2003 to March 2005
- Got a position in June 2005.

Achivements in the two years

- The laboratory was equiped:
  to develop and test detectors
- Members 1 researcher, 2 posdocs, 3 PhD students, and 1 M. Sc. student
- Construction of a electronic card to characterize the scintillators for the ACORDE detector
- Design of an emulator of signals to test the data acquisition system of ALICE
- Several simulations related with the V0 detector and the analysis of data of ALICE.
- Design of a very high momentum particle identification detector for ALICE

Guy Paic and the ICN

Silver Juchiman Award

Content

- Quantum phase transitions
- Information concepts
  - Fidelity and Fidelity Susceptibility
- Entanglement
  - Linear and von Neumann Entropies
- Conclusions

Quantum phase transitions

- Typically they are driven by purely quantum fluctuations
- Characterized by the vanishing, in the thermodynamic limit, of the energy gap
- Sudden change, non analytical, in the ground state properties of a system
- Classically they are determined by the stability properties of the potential energy surface, the order is determined by the Ehrenfest classification
- This can be extended to the quantum case: Expectation value of the Hamiltonian with respect to a variational function

Phase transitions

Family of potentials

$$V = V(x, c)\,,$$

with $x = (x_1, \cdots x_n)$ and $c = (c_1, c_2, \cdots, c_k)$.

Equilibrium and stability properties:

$$\frac{\partial V}{\partial x_j} = 0\,, \qquad \frac{\partial^2 V}{\partial x_j\, \partial x_k} > 0\,.$$

State equation: $x^{(p)} = x^{(p)}(c_1, c_2, \cdots, c_k)$

A phase transition occurs when the point $x^{(p)}(c)$ cross the separatrix of the physical system. The separatrix is the union of the bifurcation and Maxwell sets.

Separatrix

Ground state energy for a system of N particles

$$\langle H \rangle = E(x_\alpha, c_j) \rightarrow \mathcal{E} = \frac{E(x_\alpha, c_j)}{N}$$

with $\alpha = 1, \cdots n$ and $j = 1, 2, \cdots, k$.

Bifurcation and Maxwell sets: $\frac{\partial \mathcal{E}}{\partial x_k} = 0$

$$\begin{aligned} \mathcal{E}_{i,j} &= \left.\frac{\partial^2 \mathcal{E}}{\partial x_i \, \partial x_j}\right|_{x^{(p)}(c)}, \\ \mathcal{E}^{(p)} &= \mathcal{E}^{(p+1)}, \quad \left\{ \frac{\partial \mathcal{E}^{(p)}}{\partial c_j} - \frac{\partial \mathcal{E}^{(p+1)}}{\partial c_j} \right\} \delta c_j = 0\,. \end{aligned}$$

Quantum phase transitions

- A finite temperature, a quantum system is a mixture of pure states, where each one occurs with probability

$$P_k = 1/Z \exp\left(-\beta E_k\right),$$

with $\beta = \frac{1}{\kappa_B T}$ and the partition function $Z = \sum_i \exp\left(-\beta E_i\right)$.

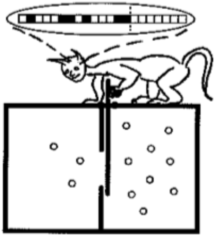
- The expectation value of an operator is given in terms of the density operator

$$\langle \hat{O} \rangle = \sum_i P_i \langle \psi_i | \hat{O} | \psi_i \rangle = \mathrm{Tr}(\rho\, \hat{O})\,.$$

- At $T = 0$ only the ground state contributes
- For $T \neq 0$, the quantum state is determined by the condition of minimum free energy instead of minimum energy.

Energy and information

Since 1961, from the Landauer principle, is known the mantra: information is physical



The reason the Maxwell demon cannot violate the second law: in order to observe a molecule, it must first forget the results of previous observations. Forgetting results, or discarding information, is thermodynamically costly ($\Delta S_e = k_B \ln 2$)

Hamiltonian Model

The Ising model for two spins $1/2$ or qubits*

$$H = \sigma_z^{(1)} \sigma_z^{(2)} + B_0 \left(\sigma_z^{(1)} + \sigma_z^{(2)} \right) ,$$

where the coupling of the qubits has been taken to be the unity. The $\sigma_z^{(i)}$ are Pauli matrices and B_0 is a magnetic field.

In terms of the total angular momentum, the Hamiltonian can be written

$$H = 2\hat{J}_z^2 - 1 + 2B_0 \hat{J}_z ,$$

where $2J_z = \sigma_z^{(1)} + \sigma_z^{(2)}$.

* J. Zhang, X. Peng, N. Rajendran, and D. Suter, Phys. Rev. Latt. **100**, 100501 (2008)

Solution

Energies and eigenstates

$$\begin{aligned} E &= \{\ -1, \quad -1, \quad 1-2B_0\,,\ 1+2B_0\}\,, \\ |\sigma_1\,,\sigma_2\rangle &= \{|+,-\rangle,\ |-,+\rangle,\quad |-,-\rangle,\quad |+,+\rangle\}\,. \end{aligned}$$

Semiclassical solution

$$H = \cos^2\theta - 2B_0\cos\theta\ ,$$

where the variational state is given by

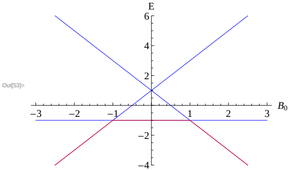
$$|j=1,\,\theta\rangle = \frac{1-\cos\theta}{2}\,|1,1\rangle + \sqrt{\frac{1-\cos^2\theta}{2}}\,|1,0\rangle + \frac{1+\cos\theta}{2}\,|1,-1\rangle\,.$$

Critical points $\theta_c\ : \{0,\ \pi,\ \arccos B_0\}$.

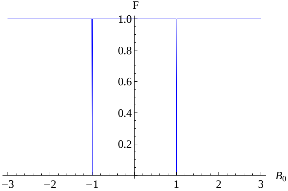
Energies and eigenstates

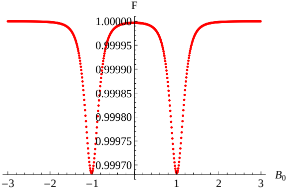
$$\begin{aligned} E &= \{1-2B_0\,,\ 1+2B_0\,,\ \ -B_0^2\,,\quad -1\}\,, \\ |\theta_c\rangle &= \{|1,-1\rangle,\qquad |1,1\rangle,\quad |1,0\rangle,\ |0,0\rangle\}\,. \end{aligned}$$

Energies and fidelity



Above in red color, the semiclassical energies and in blue color the quantum ones. Below the fidelity between the quantum solutions with B_1 and B_2. We add a probe qubit with the interaction $\epsilon\sigma_z^{(p)}(\sigma_z^{(1)}+\sigma_z^{(2)})$. Thus one has two effective Hamiltonians one with $B_1 = B_0 + \epsilon$, the other with $B_2 = B_0 - \epsilon$. At the right, we consider a small magnetic field B_x.





Fidelity

For two pure states, $\rho_1 = |\chi\rangle\langle\chi|$ and $\rho_2 = |\phi\rangle\langle\phi|$, the fidelity is defined by

$$F(|\chi\rangle\langle\chi|, |\phi\rangle\langle\phi|) = |\langle\chi|\phi\rangle|^2 ,$$

the transition probability from one state to another. Its geometric interpretation is the closeness of states.

For one mixed state ρ_2, one has

$$F(|\chi\rangle\langle\chi|, \rho_2) = \langle\chi|\rho_2|\chi\rangle ,$$

that denotes the probability to be a pure state.

For mixed states the fidelity should satisfy the properties:

$$0 \leq F(\rho_1, \rho_2) \leq 1 \quad (1)$$

$$F(\rho_1, \rho_2) = F(\rho_2, \rho_1) \quad (2)$$

$$F(U\rho_1, U\rho_2) = F(\rho_1, \rho_2) \quad (3)$$

Fidelity

Uhlmann-Jozsa proved that

$$F(\rho_1 \, , \rho_2 \,) = \left\{ \mathrm{Tr} \left(\sqrt{\sqrt{\rho_1} \, \rho_2 \, \sqrt{\rho_1}} \, \right) \right\}^2 \, ,$$

satisfies the previous properties. Another definition satisfying the same properties was given by Mendonca et al, i.e.,

$$F(\rho_1 \, , \rho_2 \,) = \mathrm{Tr}(\rho_1 \;\; \rho_2 \,) + \sqrt{1 - \mathrm{Tr}(\rho_1^2 \,)} \, \sqrt{1 - \mathrm{Tr}(\rho_2^2 \,)} \, .$$

The fidelity has a fundamental role in communication theory because measures the accuracy of a transmission.

Fidelity and Fidelity Susceptibility

The fidelity (P. Zanardi and N. Paunkovic, Phys. Rev. E 74 (2006)) can be used to determine when the ground state of a quantum system presents a sudden change as function of a control parameter. If we denote that parameter by λ one has

$$F(\lambda, \lambda + \delta\lambda) = |\langle \psi(\lambda) | \psi(\lambda + \delta\lambda) \rangle|^2 .$$

Taylor series expansion of the fidelity

$$F(\lambda_c, \lambda_c + \delta\lambda) = F(\lambda_c, \lambda_c) + \delta\lambda \left. \frac{dF}{d\lambda} \right|_{\lambda=\lambda_c} + (\delta\lambda)^2 \frac{1}{2!} \left. \frac{d^2 F}{d\lambda^2} \right|_{\lambda=\lambda_c} + \cdots ,$$

the first derivative is zero because the fidelity is a minimum and the fidelity susceptibility is defined by (W. You et al Phys. Rev. E 76 (2007))

$$\chi_F = 2 \frac{1 - F(\lambda_c, \lambda_c + \delta\lambda)}{(\delta\lambda)^2} .$$

It is dependent of the Hamiltonian term that causes the phase transition.

Entanglement

Suppose Alice and Bob are trying to create n copies of a particular bipartite state $|\Phi\rangle$, such that Alice hold the part A and Bob the part B. They are not allowed any quantum communication between them. However they have a large collection of shared singlet pairs $|\Psi_-\rangle$.
How many singlet pairs must they use up in order to create n copies of $|\Phi\rangle$?
The answer is they need to create roughly $nS_{vN}(\Phi)$, the von Neumann entropy.

Examples, the so called Bell states

$$
\begin{aligned}
|\Phi_\pm\rangle &= \frac{1}{\sqrt{2}}\Big(|+,+\rangle \pm |-,-\rangle\Big)\,, \\
|\Psi_\pm\rangle &= \frac{1}{\sqrt{2}}\Big(|+,-\rangle \pm |-,+\rangle\Big)\,.
\end{aligned}
$$

which have maximum linear and von Neumann entropies.

(a) The linear entropy is defined by $S_L = 1 - \mathrm{Tr}(\rho_2^2)$

$$\rho = \frac{1}{2}\left(|+,+\rangle\langle+,+| + |+,+\rangle\langle-,-| + |-,-\rangle\langle+,+| + |-,-\rangle\langle-,-|\right),$$

Tracing over the first subsystem one gets

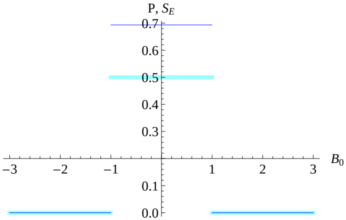
$$\rho_2 = \frac{1}{2}\left(|+\rangle\langle+| + |-\rangle\langle-|\right),$$

which implies that $S_L = 1/4$.
(b) The von Neumann entropy

$$S_{\nu N} = -\sum_k \lambda_k \ln \lambda_k$$

where λ_k denote the eigenvalues of the reduced density matrix of the subsystem 2. For the Bell state, it is immediate that $S_{\nu N} = \ln 2 = 0.693$.

Purity and von Neumann Entropy



In blue color, the von Neuman entropy and in cyan color the purity. Both as functions of the magnetic field B_0.

$$\rho_L = |+,+\rangle\langle+,+| \,, \quad \rho_M = \frac{1}{2}\left(|+,-\rangle\langle-,+| + |-,+\rangle\langle+,-|\right) \,, \quad \rho_R = |-,-\rangle\langle-,-| \,.$$

The linear entropy is defined by $P = 1 - \mathrm{Tr}(\rho_2^2)$ where $\rho_2 = \mathrm{Tr}_1(\rho_A)$ with $A = L, M,$ y R. The von Neumann entropy

$$S_{vN} = -\sum_k \lambda_k \ln \lambda_k$$

where λ_k denote the eigenvalues of the reduced density matrix ρ_2.

Hamiltonian Models

$$H = \hat{a}^\dagger \hat{a} + \omega_A \hat{J}_z + \frac{\gamma}{\sqrt{N}} \left(\hat{a}^\dagger + \hat{a}\right) \left(\hat{J}_+ + \hat{J}_-\right) \ .$$

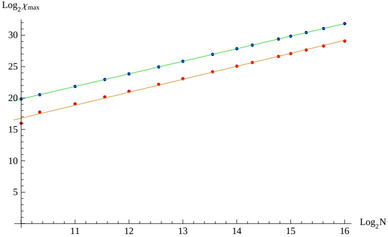
This can describe: (i) the interaction between many atoms and a single mode e.m. field of a cavity and (2) the interaction of many qubits with a single harmonic oscillator.

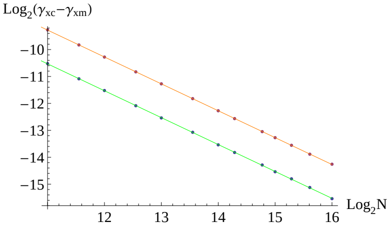
$$H = \hat{J}_z + \frac{\gamma_x}{2j-1} \hat{J}_x^2 + \frac{\gamma_y}{2j-1} \hat{J}_y^2 \ .$$

This Hamiltonian has been used to test many body approximations (LMG) or as a model of a two-mode Bose-Einstein condensate.

Scaling behavior of the fidelity susceptibility

Next, we consider $\gamma_y = -1.0$.



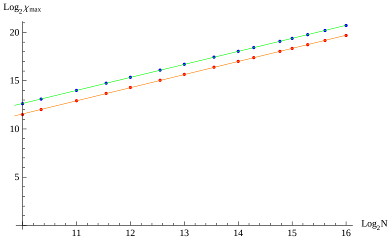


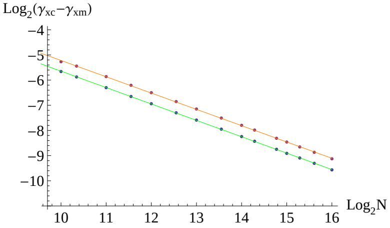
$$\chi_{max} = 2^{-0.16}\,N^2, \qquad \gamma_{xc} - \gamma_{xm} = 2^{0.46}\,N^{-1}, \quad \text{for the even case}$$
$$\chi_{max} = 2^{-2.95}\,N^2, \qquad \gamma_{xc} - \gamma_{xm} = 2^{1.71}\,N^{-1}, \quad \text{for the odd case}$$

where the thermodynamic value $\gamma_{xc} = -1$.

Scaling behavior of the fidelity susceptibility

Now, we consider $\gamma_y = -0.5.$ and the same set of number of particles mentioned before.

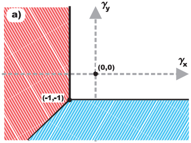


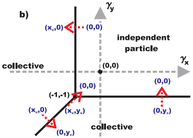


$$\chi_{max} = 2^{-0.85}\, N^{1.35}, \qquad \gamma_{xc} - \gamma_{xm} = 2^{0.87}\, N^{-0.65}, \quad \text{for the even case}$$
$$\chi_{max} = 2^{-2.04}\, N^{1.36}, \qquad \gamma_{xc} - \gamma_{xm} = 2^{1.26}\, N^{-0.65}, \quad \text{for the odd case}$$

where $\gamma_{xc} = -1.$

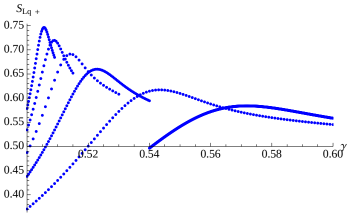
Separatrix of the LMG model

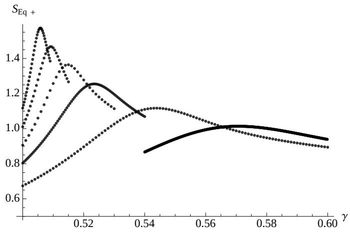




There are three regions Phys. Rev B 72 (2005); Phys. Rev B 74(2006). Phase transitions occur when one crosses these regions, we could establish the order of the phase transitions. For $\gamma_{xc} = -0.1$; one finds that $\chi_{max} \approx N^2$ and $(\gamma_{xc} - \gamma_{max}) \approx N^{-1}$. For other crossings of second order phase transitions one gets $\chi_{max} \approx N^{4/3}$ and $(\gamma_{xc} - \gamma_{max}) \approx N^{-2/3}$. The point $(-1,-1)$ is special because it has a third order phase transition ($\gamma_y = -\gamma_x - 2$).

Linear and VN entropies for the Dicke Model





At the left, the maximum values are

$$(\mathrm{N},\gamma)=\{(20,0.572),(40,0.543),(100,0.523),(200,0.514),(400,0.509),(1000,0.505)\},$$

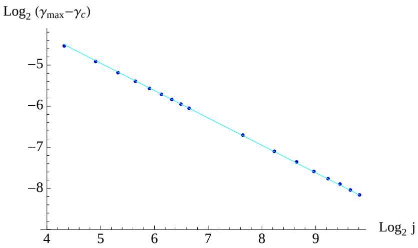
while at the right one has

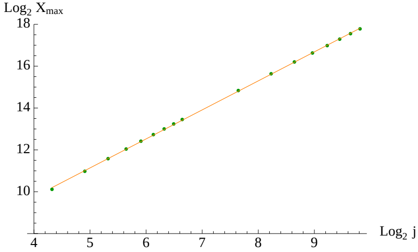
$$(\mathrm{N},\gamma)=\{(20,0.571),(40,0.544),(100,0.524),(200,0.515),(400,0.509),(1000,0.505)\}.$$

By means of the fidelity one gets

$$(\mathrm{N},\gamma)=\{(20,0.568),(40,0.543),(100,0.524),(200,0.515),(400,0.509),(1000,0.505)\}.$$

Scaling behavior





We show for the Dicke model that the coupling parameter and the maximum fidelity susceptibility also satisfy

$$(\gamma_{max} - \gamma_c) \approx N^{-\frac{2}{3}}\,, \quad \chi_{max} \approx N^{\frac{4}{3}}\,.$$

Conclusions

- Determine quantum phase crossovers, which goes to the thermodynamical limit when $N \to \infty$.
- The fidelity, fidelity susceptibility, and the linear or Von Neumann entropies give information about the quantum phase transitions for a finite number of particles, together with their scaling behavior.
- A special crossing of the triple point of the LMG model has the behavior $\chi_{max} \approx N^2$, $(\gamma_{xc} - \gamma_{max}) \approx N^{-1}$.
- Other crossings of second order phase transitions yield $\chi_{max} \approx N^{4/3}$, $(\gamma_{xc} - \gamma_{max}) \approx N^{-2/3}$. A similar behavior for the second order quantum phase transition of the Dicke model was obtained.

Thank you very much for your attention

Work done in collaboration with R. López-Peña, J. G. Hirsch, and E. Nahmad-Achar:

- PHYSICAL REVIEW B 72, 012406 (2005)
- PHYSICAL REVIEW B 74, 104118 (2006)
- Phys. Scr. 79 (2009) 065405 (14pp)
- Phys. Scr. 80 (2009) 055401 (11pp)
- Annals of Physics 325 (2010) 325344
- PHYSICAL REVIEW A 83, 051601(R) (2011)
- PHYSICAL REVIEW A 84, 013819 (2011)
- PHYSICAL REVIEW A 86, 023814 (2012)