Renormalization of a Second Order Formalism for Spin $1/2$ Fermions

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Brief Historical Review of Second Order Formalisms for spin 1/2

► (1927) V. Fock, Relativistic Quantum Mechanics of spin 1/2 through a second order differential equation.
► (1928) Dirac, P. A. M.
► (1951, 1958) Feynman - Gell-Mann\(^1\) used a two component spinorial field that satisfies \((g = 2, \xi = 0)\).

\[
[(i\partial_\mu - A_\mu)^2 + \vec{\sigma} \cdot (\vec{B} \pm i\vec{E})] \phi = m^2 \phi,
\]

Their main motivation was to describe the weak interactions.
► ...
► (1961) Hebert Pietschmann\(^2\), one loop renormalization of the Feynman-Gell-Mann theory.

Showing the equivalence with the Dirac framework has been always a goal in these works.

\(^1\)Phys. Rev. 84, 108, 1951; Phys. Rev. 109, 193, 1958
The NKR second order formalism for massive spin $3/2$ particles is an alternative to the inconsistent Rarita-Schwinger theory of electromagnetic interactions.

The case of spin $1/2$ is of interest by itself e.g. in this theory the gyromagnetic factor $g$ is a free parameter $\Rightarrow$ a low energy effective theory of particles with $g \neq 2$, e.g. proton.

We expect that this provides a better understanding of the properties of spin $1/2$ particles, e.g. the classical limit.

¿Generalizations?

In this work we used general principles of QFT to study the quantization and Renormalization. We will only compare with the conventional Dirac results only at the end.

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Quantum Fields

Quantum theories that satisfy

▶ special relativity

▶ cluster descomposition principle

can be built with quantum fields $\phi_l(x)$ defined as

$$\phi_l(x) = \int d\Gamma \left[ e^{i p \cdot x} u_l(\Gamma) a^\dagger(\Gamma) + e^{-i p \cdot x} v_l(\Gamma) a(\Gamma) \right],$$

such that under a Poincaré transformation $U(\Lambda, b)$ the fields

$$U(\Lambda, b) \phi_l(x) U(\Lambda, b)^{-1} = D(\Lambda)_{l l'} \phi_{l'}(\Lambda x + b),$$

$$[\phi_l(x), \phi_m(y)]_{\pm} = 0 \text{ for } (x - y)^2 > 0,$$

where $D(\Lambda)_{l l'}$ is a representation of $SO(3, 1)$. 
Scheme of the NKR construction of QFTs

- Spacetime Symmetries of Fields $\phi(x)$
- Second Order Equations of Motion: $[T^{\mu\nu}\partial_\mu \partial_\nu + \ldots] \phi = 0$
- Lagrangian $\mathcal{L}[\phi, \partial \phi]$
  - Noether: Poincaré Scalar
  - Hermitian
- Interactions: Minimal Coupling
  $\mathcal{L}[\phi, \partial \phi] \rightarrow \mathcal{L}[\phi, D\phi, A]$
Equations of motion of the NKR formalism

**General Idea:** To use the Poincare invariants $P^2$ and $W^2$ to construct projectors $\mathcal{P}^{(m,s)}$ over spaces of definite mass and spin. Acting these projectors on the fields results in equations of motion.

For a field $\psi^{(D,m,s)}$ with only one spin sector $s$ in a given representations $D(\Lambda)$ only a projector is necessary $\mathcal{P}^{m,s}$

$$\mathcal{P}^{m,s} = \left( \frac{P^2}{m^2} \right) \left( \frac{W^2}{-s(s+1)P^2} \right),$$

the action of this projector over the field results in the following equation of motion

$$\left( T^{D\mu\nu}_{l'l'} P_\mu P_\nu - \delta_{ll'} m^2 \right) \psi^{(D,m,s)}_{l'}(x) = 0,$$

where $T^{D\mu\nu}_{l'l'}$ is defined by $W^2 = -\frac{1}{s(s+1)} T^{D\mu\nu} P_\mu P_\nu$, it depends on the generators $M^{\mu\nu}$ of the $D(\Lambda)$. 
NKR for spin 1/2 and the representations 
\((1/2, 0) \oplus (0, 1/2)\)

For a field \(\psi^{(D,m,s=1/2)}\) in the representation \(D \equiv (1/2, 0) \oplus (0, 1/2)\) the NKR equation of motion can be deduced from the following family of hermitian Poincaré scalar Lagrangians

\[
\mathcal{L} = \partial_\mu \bar{\psi} T^{\mu\nu} \partial_\nu \psi - m^2 \bar{\psi} \psi,
\]

where \(T^{\mu\nu} = g^{\mu\nu} - igM^{\mu\nu} + \xi \gamma^5 M^{\mu\nu}\).

\(M^{\mu\nu}\) are the generators of the \((1/2, 0) \oplus (0, 1/2)\) Lorentz group representation.

\[
M^{\mu\nu} = \begin{pmatrix}
M_{(1/2,0)}^{\mu\nu} & 0 \\
0 & M_{(0,1/2)}^{\mu\nu}
\end{pmatrix}, \quad \gamma^5 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]
Finally we introduce Electromagnetic interactions are introduced through minimal coupling

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + D_\mu \bar{\psi} [g^{\mu\nu} - (ig - \xi \gamma^5) M^{\mu\nu}] D_\nu \psi - m^2 \bar{\psi} \psi, \]

\( g = 2, \xi = 0 \) corresponds to the Feynman-Gell-Mann theory.

The interactions that contains \( g \) can be rewritten as

\[ \mathcal{L}_i = -\int d^4 x e g \bar{\psi} M^{\mu\nu} \psi F_{\mu\nu}, \]

that includes the interaction \( \vec{S} \cdot \vec{B} \Rightarrow \) we recognize \( g \) as the gyromagnetic factor.
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**Feynman Rules**

\[ Z[J_\mu, \bar{\eta}, \eta] = C \int \mathcal{D}A \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[ i \int \mathcal{L}_e dx \right], \]

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + D_\mu \bar{\psi} T^{\mu \nu} D_\nu \psi - m^2 \bar{\psi} \psi + J^\mu A_\mu + \bar{\eta} \psi + \bar{\psi} \eta - \]

\[ p \quad \mu \quad q \quad \nu \]

\[ iS(p) \equiv \frac{i}{p^2 - m^2} \]

\[ i\Delta_{\mu\nu} \equiv \frac{-ig_{\mu\nu}}{q^2 + i\epsilon} \]

\[ -ieV_\mu(p, p') = -ie \left[ (p' + p)_\mu + (ig + \xi\gamma^5)M_{\mu\nu}(p' - p)^\nu \right] \]

\[ 2ie^2 g^{\mu\nu} \]
Ward Identities

As a consequence of gauge invariance there exist identities between the green functions

\[ 0 = \left[ -\frac{1}{\alpha} \Box \left( \partial_\mu \frac{\delta}{\delta J^\mu(x)} \right) - \partial_\mu J^\mu - e \left( \overline{\eta} \frac{\delta}{\delta \overline{\eta}(x)} + \eta \frac{\delta}{\delta \eta(x)} \right) \right] Z(J^\mu, \eta, \overline{\eta}) \]
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Divergencies in the Second Order Theory

Asking the Lagrangian to be dimensionless one obtains

\[ [A] = [
\psi] = 1, \]
\[ [g] = [e] = [\xi] = 0. \]

Thus the greater superficial degree of divergency of a process is

\[ D \leq 4 - F - P \]

The greater degree of divergency is:

\[ \text{quadratic for propagators} \]
\[ \text{linear for 3 lines processes e.g. } ff p \]
\[ \text{logarithmic for 4 lines processes e.g. } ff pp \]

These characteristics are necessary for a theory to be renormalizable QFT.
Free Parameters and Counterterms ($\xi = 0$)

In terms of the bare parameters $m_b^2, e_b, g_b$ the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (\partial_\mu - ie_b A_\mu) \bar{\psi} [g^{\mu\nu} - ig_b M^{\mu\nu}] (\partial_\nu + ie_b A_\nu) \psi - m_b^2 \bar{\psi} \psi.$$  

Introducing the renormalized parameters $m^2, e, g$ and the renormalized fields $A^\mu_r = Z_1^{-\frac{1}{2}} A^\mu$ $\bar{\psi}_r = Z_2^{-\frac{1}{2}} \psi$ there appear the following counterterms

$$i(p^2 - m^2) \delta_{Z_2} - i\delta_m$$

$$-i(g^{\mu\nu} q^2 - q^\mu q^\nu) \delta_{Z_1}$$

$$-ie [V^\mu(p', p)] \delta_e + egM^{\mu\nu}(p' - p)^\nu \delta_g$$

$$2ie^2 g_{\mu\nu} \delta_3$$
Dimensional Regularization

Extend the theory to $d$ dimensions. The natural objects to be extended to $d$ dimension are the Lorentz generators $M^{\mu\nu}$

\[ [M^{\alpha\beta}, M^{\mu\nu}] = -ig^{\beta\nu}M^{\alpha\mu} + ig^{\beta\mu}M^{\alpha\nu} - ig^{\alpha\mu}M^{\beta\nu} + ig^{\alpha\nu}M^{\beta\mu}, \text{ with } g^{\mu\mu} = d \]

\[ \{M^{\mu\nu}, M^{\alpha\beta}\} = \frac{1}{2}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}) - \frac{i}{2}\epsilon^{\mu\nu\alpha\beta}\gamma^5, \]

e.g. we can use the last expression to calculate to calculate a trace in a fermion loop

\[ tr\{M^{\mu\nu}M^{\alpha\beta}\} = \frac{f(d)(d)}{4}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}) \text{ with } \lim_{d \to 4} f(d) = 4, \]
Photon Propagator

As usual one can express the complete photon propagator $i\Delta^\mu_\nu (q)$ as

$$i\Delta^\mu_\nu (q) = i\Delta^\mu_\nu (q) + i\Delta^{\mu\sigma}[-i\Pi_{\sigma\rho}(q)][i\Delta^{\rho\nu}(q)] + ...$$

where $\Pi^{\mu\nu}(q)$ is the vacuum polarization

$$\Pi^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\pi(q^2),$$

Then the complete propagator is given by

$$\Delta^\mu_\nu (q) = \frac{-g^{\mu\nu} + q^\mu q^\nu \pi/q^2}{[q^2 + i\epsilon][1 + \pi]}.$$

The first condition of renormalization is that the photon doesn’t acquired mass due to the radiative corrections, i.e.

$$\pi(q^2 \to 0) = 0.$$
Vacuum polarization to one loop

It has the following contributions

\[-i(g^{\mu\nu}q^2 - q^\mu q^\nu)\pi(q)^2 = -i(g^{\mu\nu}q^2 - q^\mu q^\nu)\pi^*(q^2) - i(g^{\mu\nu}q^2 - q^\mu q^\nu)\delta Z_1\]

The first renormalization conditions requires

\[\delta Z_1 = -\pi^*(q^2 = 0),\]

Finally, imposing the renormalization condition the physical vacuum polarization is

\[\pi(q^2) = \frac{2e^2}{(4\pi)^2} \int_0^1 dx \ln \left[ \frac{m^2 - q^2x(1-x)}{m^2} \right] \left[ (1 - 2x)^2 - \frac{g^2}{4} \right],\]

for \(g = 2\) one recovers the one loop vacuum polarization of the conventional Dirac formalism.
Charge running in the Ultrarelativistic limit

Due to the quantum effects the classical Coulomb potential modifies as

$$V(\vec{x}') = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{x}} \frac{-e^2}{|\vec{q}|^2 [1 + \pi(-|\vec{q}|^2)]},$$

in the ultrarelativistic domain one has an effective charge given by

$$e_{eff}^2 = \frac{e^2}{1 + \pi(q^2 >> m^2)} = e^2 \left[ 1 - \frac{e^2}{12\pi^2} \left( 1 - \frac{3}{2} \left[ 1 - \frac{g^2}{4} \right] \right) \ln \frac{-q^2}{Am^2} \right],$$

where $A \equiv \exp \left\{ \frac{5}{3} \frac{1 - \frac{9}{5} [1 - \frac{g^2}{4}]}{1 - \frac{3}{2} [1 - \frac{g^2}{4}]} \right\}$,

Which means that the gyromagnetic factor $g$ impacts the running of the fine structure constant $\alpha(q^2)$!
Fermion Propagator

Analogously the complete fermion propagator $iS_c(p)$ could be expressed as

$$iS_c(p) = iS(p) + iS(p)[-i\Sigma(p)]iS(p) + ...$$

where $-i\Sigma(p^2)$ is the fermion self energy. Adding up the series

$$S_c(p) = \frac{1}{p^2 - m^2 - \Sigma(p) + i\epsilon},$$

Second renormalization condition: $m$ represents the physical mass of the particle, i.e. the complete propagator has a simple pole at $p^2 = m^2$

$$\Sigma(p = m^2) = 0, \quad \frac{\partial \Sigma(p)}{\partial p^2} \bigg|_{p^2=m^2} = 0.$$
Fermion Self Energy to one loop

The contributions up to one loop are

\[-i \Sigma(p^2) = -i \Sigma^*(p^2) + i(p^2 - m^2) \delta Z_2 - i \delta_m,\]

The second renormalization conditions requires

\[i \delta_m = -i \Sigma^*(p^2 = m^2) \quad \delta Z_2 = \left. \frac{\partial \Sigma^*(p)}{\partial p^2} \right|_{p^2=m^2},\]

\[
\Sigma(p^2) = \frac{\alpha}{\pi} p^2 \int_0^1 dx (x - 1) \ln \left[ \frac{m^2 x - p^2 x(1 - x)}{m^2} \right] - \frac{3\alpha m^2}{2\pi} - \frac{\alpha}{\pi} [p^2 - m^2] \int_0^1 \frac{dx}{x}.\]
**$ffp$ Vertex**

The contributions to the one particle irreducible $ffp$ vertex $\Gamma^{\mu}(q \equiv p' - p, r \equiv p' + p)$ are

$$-ie\Gamma_c^{\mu}(p', p) = -ieV^{\mu}(p', p) - ie\Gamma^{*\mu}(p', p) - ieV^{\mu}(p', p)\delta_e - ie[igM_{\mu\nu}(p' - p)^{\nu}]\delta_g,$$

where $\Gamma^{*\mu}(p^2 = p'^2 = m^2, q^2 = 0) = \frac{e^2}{(4\pi)^2} \left\{ -2\left[\frac{1}{\epsilon} - \gamma + \ln 4\pi\right] + 2\ln \frac{m^2}{\mu^2} - 4 \int \frac{dx}{x} \right\} V^{\mu}(r, q)

$$+ \left[ 2 + \left[ 1 - \frac{g^2}{4} \right] \left[\frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2}\right] \right] igM^{\mu\nu} q_{\nu} - \frac{e^2}{(4\pi m)^2} igM^{\beta\alpha} r_{\beta} q_{\alpha} r_{\mu} \right\}. $$

There is a divergency for $g \neq 2$, this can only be removed assuming that the gyromagnetic factor must be renormalized.
Renormalization of the $f\bar{f}p$ Vertex

The tensor decomposition of the sum of contributions is

$$-ie\Gamma^\mu_c(q, r) = -ieEq^\mu - ieFr^\mu - ieGigM^{\mu\nu}q_\nu - ieHiM^{\mu\nu}r_\nu$$

$$- ieHigM^{\mu\nu}r_\nu + ieJigM^{\mu\nu}r_\nu - ieIigM^{\mu\nu}q_\nu$$

Where $E, F, ..., J$ are scalar functions.

The renormalization conditions over the $f\bar{f}p$ vertex are:

- $e$ is the electric charge on mass shell, this requires that the form factor $F$ satisfies

  $$F(p^2 = p'^2 = m^2, q^2 = 0) = 1,$$

- That the effective gyromagnetic factor on mass shell is equal to $g$ plus a finite correction $\Delta g$, this requires that the form factor $G$ satisfies

  $$gG(p^2 = p'^2, q^2 = 0) = g + \Delta g,$$
Renormalized $ffp$ vertex

These renormalizations conditions determine the value of the remaining counterterms:

$$
\delta e = \frac{e^2}{(4\pi)^2} \left[ 2\left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\right) - 2 \ln \frac{m^2}{\mu^2} + 4 \int_0^1 dx/x \right],
$$

$$
\delta g = \frac{e^2}{(4\pi)^2} \left[ \frac{g^2}{4} - 1 \right] \left[ \frac{1}{\epsilon} - \gamma + \ln 4\pi - \ln \frac{m^2}{\mu^2} \right],
$$

the first expression implies $e = \sqrt{Z_1} e_d$.

Introducing these expressions one obtains the $ffp$ vertex at arbitrary momentum $(q, r)$

$$
-ie \Gamma^\mu_c(q, r) = -ie E^\mu q^\mu - ie F^\mu r^\mu - ie G g M^{\mu\nu} q_\nu - ie H g M^{\mu\nu} r_\nu
$$

$$
- ie I g M^{\beta\alpha} r^\beta q_\alpha r^\mu + J g M^{\beta\alpha} r^\beta q_\alpha q^\mu
$$
Form Factors

\[ F(r^2, q^2, r \cdot q, m) = 1 + \frac{\alpha}{4\pi} \left\{ \int_0^1 dx (2 - x) \left[ \ln \frac{\Delta_1(p, mx^{1/2}, x)}{m^2} + \ln \frac{\Delta_1(p', mx^{1/2}, x)}{m^2} \right] \\
+ \int_0^1 \int_0^{1-x} dx dy \left[ 2 \ln \frac{m^2}{\Delta_2(q, r, m, x, y)} + \frac{q^2[(\frac{q^2}{4} - 1)(x + y) + 1] + r^2[2(x + y) - (x + y)^2 - 1]}{\Delta_2(q, r, m, x, y)} \right] \\
+ \frac{r \cdot q[y - x + x^2 - y^2]}{\Delta_2(q, r, m, x, y)} + \frac{4}{(x + y)^2} \right\} \],

\[ G(r^2, q^2, r \cdot q, m) = 1 + \frac{\alpha}{4\pi} \left\{ \int_0^1 dx \left( \frac{g^2}{4} - 1 \right) \ln \frac{\Delta_1(q, m, x)}{m^2} \right. \\
+ \int_0^1 dx \left[ \ln \frac{\Delta_1(q + r, mx^{1/2}, x)}{m^2} + \ln \frac{\Delta_1(q - r, mx^{1/2}, x)}{m^2} \right] + \int_0^1 \int_0^{1-x} dy dx \frac{4dydx}{(x + y)^2} \\
+ \int_0^1 \int_0^{1-x} dy dx \left[ - 2 \ln \frac{m^2}{\Delta_2(q, r, m, x, y)} + \frac{r^2[(x + y) - 1] + (1 - \frac{q^2}{2})(r \cdot q)(y - x) + q^2}{\Delta_2(q, r, m, x, y)} \right] \left\} \right. \]
Finite correction to the gyromagnetic factor

The effective gyromagnetic factor on mass shell is given by

\[-ie\Gamma^\mu_c = -ie[\mathcal{G}(r^2 = 4m^2, q^2 = r \cdot q = 0)igM^{\mu\nu}q_\nu] + \ldots,\]

\[\mathcal{G}(r^2 = 4m^2, q^2 = r \cdot q = 0) = 1 + \frac{\alpha}{2\pi}.\]

This equation shows that the finite correction to the gyromagnetic factor to one loop is

\[\Delta g = \frac{g\alpha}{2\pi},\]

for \(g = 2\) this is just the the conventional result \(\Delta g = \frac{\alpha}{\pi}\).
Calculating the $ffpp$ vertex one observes that the divergencies are removed by the past renormalization conditions.

\[
\begin{align*}
\Lambda^{\mu\nu}_{\alpha} &= -e^4 \int\frac{dl}{(2\pi)^d} V_{\alpha}(p,p+l) dV_{\mu}(p+k+l) dV_{\nu}(p+\nu, k) \\
\Lambda^{\mu\nu}_{\alpha} &= e^4 \int\frac{dl}{(2\pi)^d} g_{\mu\nu} [p+k+l] + 4g_{\mu\nu} [p-k+l] \\
\end{align*}
\]
Perspectives

The rest of **superficially** divergent processes are (with 3 and 4 external lines)

These processes must be finite if the theory is renormalizable to one loop.

We expect that the first process to be zero due to charge conjugation symmetry.
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- We studied the one loop renormalization using path integral quantization, obtaining the Feynman rules and showing the Ward identities to all orders, they were verified to one loop.

- It was shown that the coupling constants are adimensional and that the superficial degree of divergency of a given process is bounded by the number of external lines.

- By imposing renormalization conditions (that identified the renormalized couplings) it was shown that the divergencies corresponding to the propagators, $ffp$ and $ffpp$ vertexes are removed for all $g$.

- It is remarkable that the Dirac gamma matrixes $\gamma^\mu$ are not necessary but natural objects are the Lorentz generators $M^{\mu\nu}$. 
Conclusions

- Vacuum polarizations to one loop: is gauge invariant, for \( g = 2 \) we recover the conventional result. However in general it depends on \( g \) which means that the running of the fine structure constant \( \alpha(q^2) \) depends on it. The fermion self energy is independent of \( g \) at one loop level.

- Divergencies corresponding to the \( ffp \) vertex for \( g \neq 2 \) are only removed assuming that the gyromagnetic factor must be renormalized.

- The finite correction to the gyromagnetic factor which depends on \( g \), and in the case of \( g = 2 \) one recovers the correct Schwinger correction.
Perspectives

- To finish the study of the one loop renormalization for $1/2$,
- Renormalization of the NKR formalism for spin $3/2$.
- ¿Generalizations?
Thanks
The Reduction Formula $S$

Consider the $S$ matrix elements

$$S_{\alpha\beta} = \langle k_{1}'^{\mu}, \sigma_1', ..., k_n'^{\nu}, \sigma_n'; \beta, out | p_1^\kappa, \sigma_1, ..., p_m^\theta, \sigma_m; \alpha, in \rangle$$

reduction formulas allow us to simplify

$$S_{\alpha\beta} = \frac{1}{Z_1} ... \frac{1}{Z_m} \sum_{l_i l_i'} \int \prod dx_1' ... dx_m \left[ u_{l_1'}(x_1', p_1', \sigma_1') ... u_{l_n'}(x_n', p_n', \sigma_n') \right]$$

$$\langle 0 | T(\phi_{l_1'}(x_{l_1'}), ..., \phi_{l_m}(x_{l_m})) | 0 \rangle \left[ u_{l_1}(x_1, p_1, \sigma_1) ... u_{l_m}(x_m, p_n', \sigma_m) \right]$$

- $\langle 0 | T(\phi_{l_1'}(x_{l_1'}), ..., \phi_{l_n'}(x_{l_n'}), \phi_{l_1}(x_{l_1}) ... \phi_{l_m}(x_{l_m})) | 0 \rangle$.
- $\phi_i(x_i)$ with quantum numbers $\{p_i^{\nu}, \sigma_i\}$ corresponding to in or out.
- $Z_i$ field strength renormalization of $\phi^i$.
- $u_i(x_i, p_i, \sigma_i)$ differential operators acting in $\phi_i(x_i)$.

To study the renormalization we focus on calculating $\langle 0 | T(\phi...) | 0 \rangle$. 
Free Parameters and Counterterms ($\xi = 0$)

In terms of the bare parameters $m^2_d, e_d, g_d$ the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu d} F_{d\mu\nu} + (\partial_{\mu} - ie_d A_{d\mu}) \bar{\psi}_d [g^{\mu\nu} - ig_d M^{\mu\nu}] (\partial_{\nu} + ie_d A_{d\nu}) \psi_d - m^2_d \bar{\psi}_d \psi_d.$$

Introducing the renormalized parameters $m^2_r, e_r, g_r$ and the renormalized fields $A_{\mu}^r = Z_1^{-\frac{1}{2}} A_{d\mu}$ and $\psi_r = Z_2^{-\frac{1}{2}} \psi_d$ the Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu r} F_{r\mu\nu} - \frac{1}{2} (\partial_{\mu} A_{r\mu})^2 - \frac{1}{4} F^{\mu\nu r} F_{r\mu\nu} \delta Z_1 - \frac{1}{2} (\partial_{\mu} A_{r\mu})^2 \delta Z_1$$

$$+ \partial_{\mu} \bar{\psi}_r \partial_{\mu} \psi_r - m^2_r \bar{\psi}_r \psi_r + [\partial_{\mu} \bar{\psi}_r \partial_{\mu} \psi_r - m^2 \bar{\psi}_r \psi_r] \delta Z_2 - \delta m \bar{\psi}_r \psi_r$$

$$- i e_r [\bar{\psi}_r T_{r\nu\mu} \partial_{\mu} \psi_r - \partial_{\mu} \bar{\psi}_r T_{r\mu\nu} \psi_r] A_{r\nu}^r - i e_r [\bar{\psi}_r T_{r\nu\mu} \partial_{\mu} \psi_r - \partial_{\mu} \bar{\psi}_r T_{r\mu\nu} \psi_r] A_{r\nu}^r \delta e$$

$$- i e_r [\bar{\psi}_r ( - i g_r M_{\nu\mu}) \partial_{\mu} \psi_r - \partial_{\mu} \bar{\psi}_r ( - i g_r M_{\mu\nu}) \psi_r] A_{r\nu}^r \delta g + e^2_r \bar{\psi}_r \psi_r A_{r\mu}^r A_{r\mu}^r \delta 3,$$

where

$$\delta Z_1 \equiv Z_1 - 1 \quad \delta Z_2 \equiv Z_2 - 1 \quad \delta m \equiv Z_2 [m^2_d - m^2_r],$$

$$\delta e \equiv \frac{e_d}{e_r} Z_1^{-\frac{1}{2}} Z_2 - 1 \quad \delta g \equiv \frac{e_d}{e_r} Z_1^{-\frac{1}{2}} Z_2 \left[ \frac{g_d}{g_r} - 1 \right], \quad \delta 3 \equiv \frac{e^2_d}{e_r^2} Z_1 Z_2 - 1$$
Dimensional Regulatization

We could use the conventional extension

\[ \{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu} \text{ with } g^{\mu\mu} = d, \]
\[ tr\{\gamma^\mu\} = 0, \quad tr I = f(d) \text{ with } \lim_{d \to 4} f(d) = 4, \]
\[ M^{\mu\nu} = i/4[\gamma^\mu, \gamma^\nu]. \]

but the gammas \( \gamma^\mu \) are not necessary we could use instead only the Lorentz generators \( M^{\mu\nu} \)

\[ [M^{\alpha\beta}, M^{\mu\nu}] = -ig^{\beta\nu} M^{\alpha\mu} + ig^{\beta\mu} M^{\alpha\nu} - ig^{\alpha\mu} M^{\beta\nu} + ig^{\alpha\nu} M^{\beta\mu}, \text{ con } g^{\mu\mu} = d \]
\[ \{M^{\mu\nu}, M^{\alpha\beta}\} = \frac{1}{2}(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) - \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} \gamma^5 \text{ con } tr \gamma^5 = 0, (\gamma^5)^2 = 1, \]
\[ tr M^{\mu\nu} = 0, \quad tr \{M^{\mu\nu} M^{\alpha\beta}\} = \frac{f(d)}{4}(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \text{ con } \lim_{d \to 4} f(d) = 4, \]